

Vacuum fluctuations for a charged scalar field in de Sitter spacetime with compact dimensions

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Abstract. Vacuum expectation value (VEV) of the field squared is investigated for a charged scalar field in dS spacetime with toroidally compactified spatial dimensions. Along compact dimensions general quasiperiodicity conditions are imposed and, in addition, the presence of a classical constant gauge field is assumed. The latter leads to Aharonov-Bohm type effect on the VEVs of the physical observables. The VEV of the field squared is an even periodic function of the magnetic fluxes enclosed by compact dimensions. For small values of the comoving length of a compact dimension, compared with the dS curvature scale, which corresponds to the early stages of the cosmological expansion, the contribution of the gravity in the VEV is small, and the field squared is related to the corresponding result in the Minkowski spacetime by a conformal relation. For large values of comoving length, which corresponds to the late stages of the cosmological evolution, depending on the field mass, two regimes are realized with the monotonic and oscillatory damping of the expectation value.

Keywords: de Sitter spacetime, quantum scalar field, topological Casimir effect

1. Introduction

In quantum field theory the vacuum is defined as a state of a quantized field with zero number of quanta. By taking into account that the field operator does not commute with the operator of the number of quanta, from here it follows that in the vacuum state the field is subject to quantum fluctuations. These fluctuations are similar to those for the coordinate and momentum in the ground state of a harmonic oscillator in quantum mechanics.

The properties of vacuum fluctuations of a quantum field depend on the geometry and topology of the background spacetime. In particular, the presence of boundaries gives rise to the modification of the spectrum. As a consequence of that the vacuum expectation values (VEVs) of physical observables are shifted by an amount depending on the geometry of boundaries and on boundary conditions imposed on the field operator. This is the well-known Casimir effect (for reviews see [1]). A similar effect, referred as topological Casimir effect, arises when the boundary conditions originate from the nontrivial topology of the spacetime. A number of fundamental theories are formulated in backgrounds with nontrivial topology (Kaluza-Klein theories, supergravity, string theories) and it is of interest to investigate the properties of the vacuum state for the corresponding geometries.

In the present paper we consider a problem with two sources of the vacuum polarization for a charged scalar field. The first one corresponds to the curvature of the background spacetime and the second one is related to the nontrivial topology. In order to have an exactly solvable problem we take a highly symmetric geometry with the local geometrical characteristics being the same as those for the de Sitter (dS) spacetime and with a part of spatial dimensions compactified on a torus. The dS spacetime is among the most important background geometries in cosmology. A large number of models for the early expansion of the Universe are based on the assumption that the Universe has gone through an accelerated expansion phase (inflation) described by the dS geometry [2]. From the other side, recent observations of high-redshift supernovae, galaxy clusters and cosmic microwave background [3] indicate that the Universe is accelerating at the present epoch and the corresponding expansion is well approximated by a model with a positive cosmological constant as a dominant source. In this case, the standard cosmology leads to dS geometry as a future attractor for the Universe expansion.

Motivated by the importance of the dS geometry in gravitational physics, in [4,5,6,7] the properties of the vacuum state for untwisted and twisted scalar and fermionic fields induced by the toroidal compactification of spatial dimensions in dS spacetime were investigated. As local characteristics of the vacuum state the VEVs for the field squared and the energy-momentum tensor were considered. For charged fields another important bilinear characteristic is the VEV of the current density. For the same geometry of locally dS spacetime with compact dimensions, this VEV in the cases of the charged scalar and fermionic fields have been investigated in [8]. The VEV of the energy-momentum tensor is discussed in [9]. Here we consider the VEV of the field squared for a charged scalar field with quasiperiodicity conditions along compact dimensions and in the presence of an external constant gauge field.

The paper is organized as follows. In the next section the geometry of the problem and the boundary conditions are specified. The section 3 is devoted to the investigation of the VEV of the field squared. The main results of the paper are summarized in section 4.

2. Problem setup

The background geometry we are going to consider is described by the line element

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{l=1}^D (dx^l)^2, \quad (2.1)$$

with the spacetime coordinates $(x^0 = t, x^1, \dots, x^D)$ and with D being the number of spatial dimensions. The line element (2.1) coincides with that for dS spacetime generated by the positive cosmological constant $\Lambda = D(D-1)\alpha^{-2}/2$, where α is the curvature radius. However, the global properties of the manifold under consideration will be different from those for dS geometry. For the coordinates $\mathbf{x}_p = (x^1, \dots, x^p)$, as usual, we will take $-\infty < x^l < \infty$, $l = 1, \dots, p$, whereas the remaining coordinates $\mathbf{x}_q = (x^{p+1}, \dots, x^D)$, with $q = D - p$, are compactified to a torus $T^q = (S^1)^q$ with the lengths of the compact dimensions L_l : $0 \leq x^l \leq L_l$, $l = p+1, \dots, D$. This means that in our geometry for the spatial topology one has $R^p \times T^q$.

We are interested in the change of the properties of the vacuum state for a charged scalar field $\varphi(x)$, induced by the curvature and nontrivial topology. Assuming a general value for the curvature coupling parameter ξ , the field equation reads

$$(D_\mu D^\mu + m^2 + \xi R)\varphi(x) = 0, \quad (2.2)$$

where $R = D(D+1)/\alpha^2$ is the Ricci scalar of the background spacetime. We assume the presence of a classical abelian gauge field A_μ and the gauge extended covariant derivative operator is given by $D_\mu = \nabla_\mu + ieA_\mu$, with e being the charge of the field quanta. The topology is nontrivial and it is required to specify the periodicity conditions for the field operator along compact dimensions. We consider generic quasiperiodic boundary conditions with general phases α_l , $l = p+1, \dots, D$:

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\alpha_l} \varphi(t, \mathbf{x}_p, \mathbf{x}_q), \quad (2.3)$$

where $\mathbf{e}_l$ is the unit vector along the dimension x^l . The special cases $\alpha_l = 0$ and $\alpha_l = \pi$ correspond to untwisted and twisted scalar fields, respectively.

For the gauge field we will assume that $A_\mu = \text{const}$. Though the corresponding field tensor vanishes, the nontrivial topology of the background spacetime gives rise to the Aharonov-Bohm type effect on the VEVs of physical observables. In this simple configuration, by the gauge transformation $A'_\mu = A_\mu + \partial_\mu \omega$, $\varphi'(x) = e^{-ie\omega} \varphi(x)$, with the function $\omega = -A_\mu x^\mu$, the gauge field is excluded from the field equation (in the new gauge $A'_\mu = 0$). However, the gauge field appears in the quasiperiodicity conditions for the transformed field:

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l \mathbf{e}_l) = e^{i\tilde{\alpha}_l} \varphi'(t, \mathbf{x}_p, \mathbf{x}_q), \quad (2.4)$$

with the new phases

$$\tilde{\alpha}_l = \alpha_l + eA_l L_l. \quad (2.5)$$

The physical results will depend on the phases in the periodicity conditions and on the gauge potential in the form of the combination (2.5). The shift in the phases due to the gauge field may be presented as $eA_l L_l = -2\pi\Phi_l / \Phi_0$, where Φ_l is the flux enclosed by the circle corresponding to the l th compact dimension and $\Phi_0 = 2\pi / e$ is the flux quantum. The consideration below will be presented in terms of the gauge transformed field $\varphi'(x)$ omitting the prime. In the new gauge $D'_\mu = \nabla_\mu$.

In the problem under consideration the properties of the vacuum state are encoded in two-point functions. In particular, from the point of view of the evaluation of the VEVs the Hadamard function is of special interest. The latter is defined as the VEV

$$G^{(1)}(x, x') = \langle 0 | \varphi(x) \varphi^\dagger(x') + \varphi^\dagger(x') \varphi(x) | 0 \rangle, \quad (2.6)$$

where $|0\rangle$ stands for the vacuum state and the dagger corresponds to the hermitian conjugate. The expression for the Hadamard function has been derived in [8] and we use it for the investigation of the VEV for the field squared.

3. VEV of the field squared

Having the Hadamard function, the VEV of the field squared is evaluated as the coincidence limit

$$\langle \varphi \varphi^\dagger \rangle = \frac{1}{2} \lim_{x' \rightarrow x} G^{(1)}(x, x'). \quad (3.1)$$

In this limit the two-point function is divergent and some renormalization procedure is needed. The important point here is that the local geometry is not changed by the toroidal compactification and the divergences are the same as in the uncompactified dS spacetime. As a consequence, the renormalization is reduced to the one in the absence of the compactification.

By using the formula for the Hadamard function from [8], the VEV of the field square is decomposed as

$$\langle \varphi \varphi^\dagger \rangle_{p,q} = \langle \varphi \varphi^\dagger \rangle_{p+1,q-1} + \Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q}, \quad (3.2)$$

where $\langle \varphi\varphi^\dagger \rangle_{p+1,q-1}$ is the VEV in dS spacetime with topology $R^{p+1} \times T^{q-1}$ which is obtained from the topology $R^p \times T^q$ uncompactifying the r -th dimension. The contribution $\Delta_r \langle \varphi\varphi^\dagger \rangle_{p,q}$ is induced by the compactification of the r -th dimension. It is presented in the form

$$\Delta_r \langle \varphi\varphi^\dagger \rangle_{p,q} = \frac{4\eta^D L_r}{(2\pi)^{(p+3)/2} V_q \alpha^{D-1}} \int_0^\infty dz z [I_{-\nu}(\eta z) + I_\nu(\eta z)] K_\nu(\eta z) \times \sum_{n=1}^\infty \frac{\cos(n\tilde{\alpha}_r)}{(nL_r)^{p-1}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p-1)/2}(nL_r \sqrt{z^2 + u_r^2}), \quad (3.3)$$

where $V_q = L_{p+1} \dots L_D$ is the volume of the compact subspace, $I_\nu(x)$, $K_\nu(x)$ are the modified Bessel functions of the order

$$\nu = \left[D^2 / 4 - D(D+1)\xi - m^2 \alpha^2 \right]^{1/2}, \quad (3.4)$$

and $f_\mu(x) = x^\mu K_\mu(x)$. The variable η is related to the time coordinate as $\eta = |\tau|$, where

$\tau = -\alpha e^{-t/\alpha}$ is the conformal time. In (3.3) we have defined $u_r^2 = \sum_{l=p+1, \neq r}^D k_l^2$, where

$$k_l = (2\pi n_l + \tilde{\alpha}_l) / L_l, \quad n_l = 0, \pm 1, \pm 2, \dots, \quad (3.5)$$

is the component of the momentum along the l -th compact dimension. The summation goes over $\mathbf{n}_{q-1}^{(r)} = (n_{p+1}, \dots, n_{r-1}, n_{r+1}, \dots, n_D)$.

After the recurring application of (3.2), the VEV can be presented in the form

$$\langle \varphi\varphi^\dagger \rangle_{p,q} = \langle \varphi\varphi^\dagger \rangle_{dS} + \langle \varphi\varphi^\dagger \rangle_c, \quad \langle \varphi\varphi^\dagger \rangle_c = \sum_{l=1}^q \Delta_{D-l+1} \langle \varphi\varphi^\dagger \rangle_{D-l,l}, \quad (3.6)$$

where $\langle \varphi\varphi^\dagger \rangle_{dS}$ is the VEV in the uncompactified dS spacetime and $\langle \varphi\varphi^\dagger \rangle_c$ is induced by the compactification to a torus. The divergences in the coincidence limit are contained in the part $\langle \varphi\varphi^\dagger \rangle_{dS}$ and the renormalization is required for that part only. The topological part $\langle \varphi\varphi^\dagger \rangle_c$ and the separate contributions $\Delta_r \langle \varphi\varphi^\dagger \rangle_{p,q}$ are even periodic functions of the magnetic fluxes Φ_l with the period equal to the flux quantum. In the model with a single compact dimension x^D ($q=1, p=D-1$) the expression for the topological part is simplified to

$$\langle \varphi\varphi^\dagger \rangle_c = \frac{4\alpha^{1-D} \eta^D}{(2\pi)^{D/2+1}} \int_0^\infty dz z [I_{-\nu}(\eta z) + I_\nu(\eta z)] K_\nu(\eta z) \sum_{n=1}^\infty \frac{\cos(n\tilde{\alpha}_D)}{(nL_D)^{D-2}} f_{D/2-1}(nL_D z). \quad (3.7)$$

In the numerical examples below we will consider this case.

For a conformally coupled massless scalar field the curvature coupling parameter is given by $\xi = (D-1)/4D$ and one gets $\nu = \sqrt{1/4 - m^2 \alpha^2}$. For a massless field this gives $\nu = 1/2$ and the modified Bessel functions are expressed in terms of the elementary functions. For the

combination in (3.3) we have $[I_{-1/2}(x) + I_{1/2}(x)]K_{1/2}(x) = 1/x$. The integral is evaluated by using the formula

$$\int_0^\infty du f_\mu(a\sqrt{u^2 + b^2}) = \frac{1}{a} \sqrt{\frac{\pi}{2}} f_{\mu+1/2}(ab), \quad (3.8)$$

and we find

$$\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q} = \frac{2(\eta/\alpha)^{D-1}}{(2\pi)^{p/2+1} V_q L_r^{p-1}} \sum_{n=1}^\infty \frac{\cos(n\tilde{\alpha}_r)}{n^p} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{p/2}(nL_r u_r). \quad (3.9)$$

Note that in terms of the conformal time coordinate τ one has $ds^2 = (\alpha/\tau)^2 ds_M^2$, where ds_M^2 is the Minkowskian line element in Cartesian coordinates. From here it follows that for a conformally coupled scalar field the problem on dS bulk is conformally related to the problem on the locally Minkowski bulk with spatial topology $R^p \times T^q$. The result could be obtained from the corresponding result in the Minkowski bulk (denoted here by $\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q}^{(M)}$) by using the standard relation $\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q} = (\eta/\alpha)^{D-1} \Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q}^{(M)}$. For a single compact dimension x^D from (3.9) one gets

$$\langle \varphi \varphi^\dagger \rangle_c = \frac{2(\eta/\alpha L_D)^{D-1}}{(2\pi)^{(D+1)/2}} \sum_{n=1}^\infty \frac{\cos(n\tilde{\alpha}_D)}{n^{D-1}}. \quad (3.10)$$

Now let us consider the behavior of the topological part $\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q}$ in the asymptotic regions of the ratio L_r/η . Note that the comoving length of the r th compact dimensions is given by $\alpha L_r/\eta$. Hence, the ratio L_r/η is the comoving length of the compact dimension measured in units of the curvature radius α . For small values of this ratio, $L_r/\eta \ll 1$, we introduce a new integration variable $y = L_r z$. The arguments of the modified Bessel functions are large and to the leading order one gets $[I_{-\nu}(x) + I_\nu(x)]K_\nu(x) \approx 1/x$. As a consequence, the leading order term in the asymptotic expansion coincides with the Minkowskian VEV multiplied by the conformal factor $(\eta/\alpha)^{D-1}$:

$$\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q} \approx (\eta/\alpha)^{D-1} \Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q}^{(M)}, \quad L_r/\eta \ll 1. \quad (3.11)$$

Hence, in the limit under consideration the effects of curvature are small. Note that for fixed values of L_r/α the limit under consideration corresponds to the early stages of the cosmological expansion ($t \rightarrow -\infty$), and the topological part behaves like $e^{-(D-1)t/\alpha}$. By taking into account that the part $\langle \varphi \varphi^\dagger \rangle_{dS}$ is time independent, we conclude that in the early stages of the cosmological expansion the topological part dominates in the VEV of the field squares.

For large values of L_r/η , again, we introduce a new integration variable $y = L_r z$. Two cases should be considered separately. For real values of the parameter ν , we use the formulas for the modified Bessel functions for small values of the argument:

$$[I_{-\nu}(x) + I_{\nu}(x)]K_{\nu}(x) \approx \frac{1}{2} \frac{(2/x)^{2\nu} \Gamma(\nu)}{\Gamma(1-\nu)}. \quad (3.12)$$

After the evaluation of the integral over y we find

$$\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q} \approx \frac{2^{\nu-(1+p)/2} \eta^{D-2\nu} \Gamma(\nu)}{\pi^{(p+3)/2} V_{q-1} \alpha^{D-1}} \sum_{n=1}^{\infty} \frac{\cos(n\tilde{\alpha}_r)}{(nL_r)^{p+1-2\nu}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+1)/2-\nu}(nL_r u_r), \quad (3.13)$$

where $V_{q-1} = V_q / L_r$. For fixed values of the ratio L_r / α , the limit under consideration corresponds to late stages of the cosmological evolution ($t \rightarrow +\infty$), and the topological part is suppressed by the factor $\exp[-(D-2\nu)t / \alpha]$. Hence, in this limit the total VEV is dominated by the uncompactified dS part $\langle \varphi \varphi^\dagger \rangle_{dS}$.

In the same limit $L_r / \eta \gg 1$, for pure imaginary values of the parameter ν , by taking into account that for small values of x one has

$$[I_{-\nu}(x) + I_{\nu}(x)]K_{\nu}(x) \approx \text{Re} \left[\frac{(2/x)^{2i|\nu|} \Gamma(i|\nu|)}{\Gamma(1-i|\nu|)} \right], \quad (3.14)$$

for the field squared we find the asymptotic behavior

$$\begin{aligned} \Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q} &\approx \frac{4\eta^D \alpha^{1-D}}{(2\pi)^{(p+3)/2} V_{q-1}} \sum_{n=1}^{\infty} \sum_{\mathbf{n}_{q-1}^{(r)}} \frac{\cos(n\tilde{\alpha}_r)}{(nL_r)^{p+1}} \\ &\times \text{Re} \left[2^{i|\nu|} \Gamma(i|\nu|) (nL_r / \eta)^{2i|\nu|} f_{(p+1)/2-i|\nu|}(nL_r u_r) \right]. \end{aligned} \quad (3.15)$$

Defining B_s and ϕ_s by the relation

$$B_s e^{i\phi_s} = 2^{i|\nu|} \Gamma(i|\nu|) \sum_{n=1}^{\infty} \frac{\cos(n\tilde{\alpha}_r)}{n^{p+1-2i|\nu|}} \sum_{\mathbf{n}_{q-1}^{(r)}} f_{(p+1)/2-i|\nu|}(nL_r u_r), \quad (3.16)$$

we can write this formula in terms of the synchronous time:

$$\Delta_r \langle \varphi \varphi^\dagger \rangle_{p,q} \approx \frac{4\alpha B_s e^{-Dt/\alpha}}{(2\pi)^{(p+3)/2} V_{q-1} L_r^{p+1}} \cos[2|\nu|t/\alpha + 2|\nu| \ln(L_r / \alpha) + \phi_s]. \quad (3.17)$$

Hence, in the case under consideration at late stages of the cosmological evolution the topological part is suppressed by the factor $e^{-Dt/\alpha}$ and the damping of the corresponding VEV has an oscillatory nature.

In figure 1 we have plotted the dependence of the topological contribution in the VEV of the field squared on the phase in the periodicity condition for a conformally coupled field. The graph is displayed for the model $D=4$ with a single compact dimension of the length L_r with $L_r / \eta = 0.5$. The numbers near the curves are the values of the product $m\alpha$. The latter is the field mass measured in units of the curvature scale $1/\alpha$. The topological part is an even periodic function of $\tilde{\alpha}_r$ with the period 2π and the graphs are plotted for one period.

Figure 2 presents the topological part in the VEV of the field squared versus the proper length of the compact dimension, measured in units of the curvature radius. The graphs are plotted for $m\alpha = 0.25, 2$ (numbers near the curves) and for $\tilde{\alpha}_r = \pi/2$. For $m\alpha = 2$ the parameter is pure imaginary and the behavior for large L_r/η is oscillatory damping, in accordance with the asymptotic analysis given above.

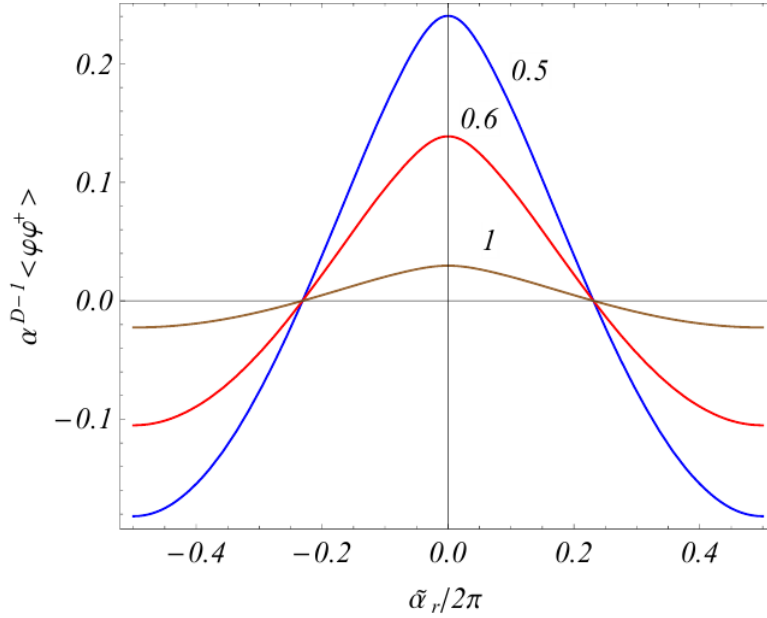


Figure 1: The topological part in the VEV of the field squared as a function of the phase $\tilde{\alpha}_r$ in the model $D=4$ with a single compact dimensions for a conformally coupled field. The graphs are plotted for fixed values of $m\alpha$ (numbers near the curves) and for $L_r/\eta = 0.5$.

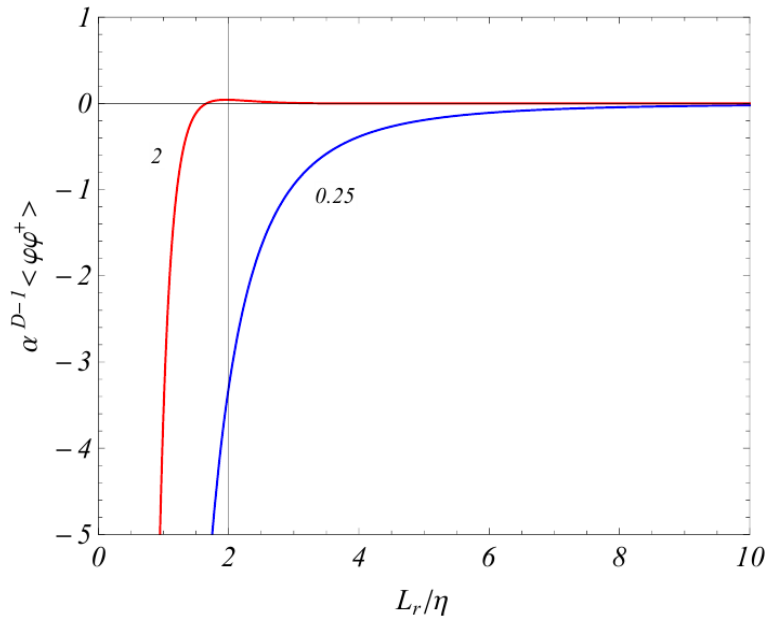


Figure 2: The topological part in the VEV of the field squared as a function of the $m\alpha$ and the graphs are plotted for fixed $\tilde{\alpha}_r = \pi/2$.

4. Conclusion

Thus, we have investigated the effects of curvature and of nontrivial topology on the vacuum state for a charged scalar field. The background geometry is locally dS and the nontrivial topology is induced by compactification of a part of spatial dimensions to a torus. Along compact dimensions general quasiperiodicity conditions are imposed and, in addition, the presence of a constant gauge field is assumed. The high symmetry of the background geometry allows to obtain closed expressions for the VEVs of physical observables. As such we have considered the VEV of the field squared. It is decomposed into the dS and topological parts. The toroidal compactification does not change the local geometry and the renormalization is required for pure dS part only. The topological part is an even periodic function of the magnetic fluxes enclosed by compact dimensions with periods equal to the flux quantum. For a conformally coupled massless field the topological part is conformally related to the corresponding quantity for the locally Minkowskian bulk with compact dimensions.

For small values of the proper lengths of compact dimensions, compared with the curvature radius, the effects of the gravitational field on the topological contribution in the VEV of the field squared are weak and, again, one has a conformal relation with the Minkowskian result. The effects of curvature are essential when the proper lengths of compact dimensions are large compared with the curvature radius. In this range, the asymptotic behavior of the topological contribution is qualitatively different for real and imaginary values of the parameter ν . In the first case the damping of the topological part, as a function of the lengths (or as a function of the time t), is monotonic. For imaginary ν , the damping is oscillatory, given by (3.15) or by (3.17) in terms of the synchronous time.

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References

- [1] V.M. Mostepanenko, N.N. Trunov, *The Casimir Effect and Its Applications* (Clarendon, Oxford, 1997); K.A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002); M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, *Advances in the Casimir Effect* (Oxford University Press, Oxford, 2009); *Lecture Notes in Physics: Casimir Physics*, Vol. 834, edited by D. Dalvit, P. Milonni, D. Roberts, F. da Rosa (Springer, Berlin, 2011).
- [2] A.D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic Publishers, Chur, Switzerland 1990).
- [3] A.G. Riess et al., *Astrophys. J.* **659** (2007) 98; D.N. Spergel et al., *Astrophys. J. Suppl. Ser.* **170** (2007) 377; U. Seljak, A. Slosar, P. McDonald, *JCAP* **0610** (2006) 014; E. Komatsu et al., arXiv:0803.0547.
- [4] A.A. Saharian, M.R. Setare, *Phys. Lett. B* **659** (2008) 367.
- [5] S. Bellucci, A.A. Saharian, *Phys. Rev. D* **77** (2008) 124010.
- [6] A.A. Saharian, *Class. Quantum Grav.* **25** (2008) 165012.
- [7] E.R. Bezerra de Mello, A.A. Saharian, *J. High Energy Phys.* **04** (2009) 046.
- [8] S. Bellucci, A.A. Saharian, H.A. Nersisyan, *Phys. Rev. D* **88** (2013) 024028.
- [9] E.V. Chubaryan, A.S. Kotanjyan, A.A. Saharian, D.H. Simonyan, *Grav. Cosm.* **22** (2016) 187.