Armenian Journal of Physics, 2019, vol. 12, issue 1, pp. 119-123

Invariant Solution of Dirac Equation in Crossed Electric and Magnetic Fields. Spin Integral of Motion. (*H* < *E*)

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Received 15 March 2019

Abstract. In this paper analytic and invariant solution of the electron motion in crossed electric and magnetic fields where $H < E$ is examined. It is shown that the invariant solution of the proposed problem is formally similar to that of quantum harmonic oscillator. Examining the polarity operator in external field, it is deduced that there is an expression consisting of spin components which remains constant throughout the motion and therefore is integral of motion. Eigenvalues and eigenfunctions corresponding to spin integral of motion are found and analyzed.

Keywords: Dirac equation, spin integral, integral of motion, crossed fields, invariant solution

1. Introduction

The problem of relativistic electron in external field has always been very interesting research area. Volkov [1] solved the problem of electron in the field of electromagnetic plane wave. Since then many authors examined the problem of relativistic electron in different electromagnetic field configurations. Of particular interest is the configuration where the absolute value of electric and magnetic fields are not equal. This problem was examined in [2], [3], but the solutions in both cases were not represented in invariant form.

The crossed field configuration is very interesting topic recently. Non-commutativity effects in the Dirac equation in crossed fields are discussed in [4-8]. Also, crossed field configuration is widely used when examining the Dirac equation in graphene under electric and magnetic fields (see [9], [10]). So, invariant solution of Dirac equation can be widely used in applications of other problems with similar configuration of fields.

In this paper, the motion of a particle in crossed electric (***E***) and magnetic (***H***) fields where ***H*** < ***E*** is examined. The invariant solution of the proposed problem is found. In addition, the spin integrals of motion for relativistic electron in crossed field are found.

2. Theoretical Approach

Without loss of generality, we take the frame where the electric field is directed along the ***y*** axis and the magnetic field is directed along the ***z*** axis. The crossed field is described by the four-vector potential which satisfies the Lorenz gauge condition

$$A^{\mu} = a^{\mu}\phi \ , \tag{1}$$

where $\phi = k^{\mu}x_{\mu} = \left(t - \frac{xn}{c}\right)$. k^{μ} is the four-wave vector and is written as follows

$$k^{\mu} = \frac{1}{c}(1, n, 0, 0), \ n = \frac{H}{E}. \tag{2}$$

Klein-Gordon: Firstly, we examine the problem for zero-spin particles. The equation that describes the motion of fermions is the Klein-Gordon equation which has the form

$$\left[\left(p^{\mu}-\frac{e}{c}A^{\mu}\right)^{2}-m^{2}c^{2}\right]\psi=0. \tag{3}$$

Following the similar procedures of order reduction and other mathematical manipulations of Klein-Gordon equation that are thoroughly described in [4], we get the following equation

$$U''+\left[\frac{1}{\hbar^{2}k^{2}}\left(\frac{\left(a^{\mu}p_{\mu}\right)^{2}}{a^{2}}+\frac{\left(p^{\mu}k_{\mu}\right)^{2}}{k^{2}}\right)+\frac{e^{2}a^{2}}{\hbar^{2}k^{2}c^{2}}v^{2}\right]U=0. \tag{4}$$

After introducing measureless quantities as follows

$$z=\alpha v,\ a_{0}=\frac{1}{\hbar^{2}k^{2}}\left(\frac{a^{\mu}p_{\mu}}{a^{2}}+\frac{p^{\mu}k_{\mu}}{k^{2}}\right),\ b_{0}=\frac{e^{2}a^{2}}{\hbar^{2}k^{2}c^{2}} \tag{5}$$

Considering that $\boldsymbol{H}<\boldsymbol{E} \Rightarrow n^{2}-1<0$, for the final equation we get

$$U''+\left(\gamma+z^{2}\right)U=0, \tag{6}$$

where $\gamma=\frac{a_{0}}{\sqrt{b_{0}}}$. The solution of (6) is given by degenerate hyper geometric function, in particular with the parabolic cylinder function

$$U=D_{-\frac{1+i\sqrt{\gamma}}{2}}\left((1+i)\xi\right)=\frac{\exp(-i\xi^{2}/2)}{\Gamma\left(1+\sqrt{\gamma}\right)}\int_{0}^{\infty}\exp(-(1+i)\xi x-x^{2}/2)x^{-\frac{i\sqrt{\gamma}-1}{2}}dx. \tag{7}$$

It is easy to verify that for all the values of γ, wave functions satisfy the standard conditions. Therefore, the specter of energy is continuous and energy can have arbitrary values from $-\infty$ to ∞.

Dirac: Now let us proceed to the solution of Dirac equation which describes the motion of the relativistic electron is external electromagnetic field. Dirac equation has the form

$$\left[\left(p^{\mu}-\frac{e}{c}A^{\mu}\right)^{2}-m^{2}c^{2}-\frac{i\hbar e}{2c}F_{\mu\nu}\sigma^{\mu\nu}\right]\psi=0, \tag{8}$$

where $F_{\mu\nu}$ is the tensor of electromagnetic field. Following the similar procedures and manipulations of Dirac described in the corresponding section of [4], we finally get the equation

$$U'' + \left(\gamma' + \xi^2\right)U = 0, \tag{9}$$

where

$$\gamma' = \frac{a_1}{\sqrt{b_1}},\ \lambda_{12} = p^2 - \frac{m^2c^2}{\hbar^2} \pm \frac{e\hbar}{c}\sqrt{H^2 - E^2}, \tag{10}$$

$$a_1 = a_0 - \frac{\lambda}{\hbar^2 k^2},\ b_1 = b_0 \tag{11}$$

Similarly, the solution of (9) is the parabolic cylinder function which is formally the same as (7)

$$U = D_{-\frac{1+i\sqrt{\gamma'}}{2}}\left(\left(1+i\right)\xi\right) = \frac{\exp(-i\xi^2/2)}{\Gamma\left(1+\sqrt{\gamma'}\right)}\int_0^\infty \exp(-\left(1+i\right)\xi x - x^2/2)x^{-\frac{i\sqrt{\gamma'}-1}{2}}dx.$$

The specter of energy is continuous.

Spin Integral of Motion: Finally, let us examine the problem of determining the spin integrals of motion. Polarity operator of the free particle in external electromagnetic field has the following form

$$T^\mu = \frac{\gamma^5\left(m\gamma^\mu - p^\mu\right)}{m}\ . \tag{12}$$

Note that operator (12) is commutative with the Dirac equation for relativistic electron. Covariant form of the polarity operator in external electromagnetic field is written by substituting $p^\mu \to P^\mu$

$$T^\mu = \frac{\gamma^5\left(\gamma^\mu m - P^\mu\right)}{m}, \tag{13}$$

where $P^\mu = p^\mu - eA^\mu$, or in normal form

$$\vec{T} = \gamma^0\vec{\Sigma} - \frac{\gamma^5\vec{P}}{m}, T^0 = \frac{\vec{\Sigma}\vec{P}}{m}. \tag{14}$$

Considering the following equations

$$\begin{aligned} \frac{dT^\mu}{dt} &= \frac{\partial T^\mu}{\partial t} + \left(\frac{1}{i\hbar}\right)\left[T^\mu, H^F\right], \\ \frac{\partial T^\mu}{\partial t} &= \left(\frac{1}{m}\right)\gamma^5 e\left(\frac{\partial A^\mu}{\partial t}\right), \end{aligned} \tag{15}$$

where $\boldsymbol{H^F}$ is the Hamiltonian of the system.

$$H^F = \gamma^0 \vec{\gamma}\left(-i\nabla - e\vec{A}\right) + \gamma^0 m + e\phi. \tag{16}$$

From equations (15) and (16) we get

$$\frac{dT^\mu}{dt} = -e\gamma^5\gamma^0\gamma_\nu \frac{F^{\mu\nu}}{m}, \tag{17}$$

or in normal form

$$\frac{d\vec{T}}{dt} = \frac{e}{m}\left[\left[\Sigma H\right] - \gamma^5 \vec{E}\right],\ \frac{d\overrightarrow{T_0}}{dt} = \frac{e}{m}\vec{\Sigma}\vec{E}. \tag{18}$$

Now, using the results above, let us proceed to our proposed problem. From the initial configuration of the fields, one can easily verify the following equations

$$\frac{d\overrightarrow{T_x}}{dt} = \frac{e}{m}\left(\Sigma_y H_z - \Sigma_z H_y - \gamma^5 E_x\right) = \frac{e}{m}\Sigma_y H, \tag{19}$$

$$\frac{d\overrightarrow{T_y}}{dt} = \frac{e}{m}\left(\Sigma_z H_x - \Sigma_x H_z - \gamma^5 E_y\right) = \frac{e}{m}\Sigma_x H - \gamma^5 E, \tag{20}$$

$$\frac{d\overrightarrow{T_z}}{dt} = \frac{e}{m}\left(\Sigma_x H_y - \Sigma_y H_x - \gamma^5 E_z\right) = 0, \tag{21}$$

$$\frac{d\overrightarrow{T_0}}{dt} = \frac{e}{m}\left(\Sigma_x E_x + \Sigma_y E_y + \Sigma_z E_z\right) = \frac{e}{m}\Sigma_y E, \tag{22}$$

From (19), (22) and (2) we can write the following system of equations

$$\begin{cases} \dfrac{dT_x}{dt} = -\dfrac{e}{m}\Sigma_y H \\ \dfrac{dT_0}{dt} = \dfrac{e}{m}\Sigma_y E \end{cases} \Rightarrow \begin{cases} \dfrac{dT_x}{dt} = -\dfrac{en}{m}\Sigma_y E \\ \dfrac{dT_0}{dt} = \dfrac{e}{m}\Sigma_y E \end{cases} \tag{23}$$

By subtracting the equations of the system (23), we get

$$\frac{dT_x}{dt} = -n\frac{dT_0}{dt} \Rightarrow \frac{d\left(T_x - nT_0\right)}{dt} = 0 \Rightarrow T_x - nT_0 = const \tag{24}$$

Finally, we have that the expression $T_x - nT_0$ consisting of spin operator components remains constant during the motion. Hence it is integral of motion.

3. Results and Discussion

As we saw this particular research is the continuation of the research [4]. Although the formal mathematical procedures of the solution are similar, the result is dramatically different in the sense of the energy specter and the class of solutions. So, the invariant solution is found for the proposed problem in case when $H < E$. In addition, integral of motion is examined and found exactly. It is shown that the expression $nT^0 - T_x$ is remaining constant throughout the motion of relativistic electron in crossed field.

Acknowledgements

The authors wish to thank to Dr. A. Kotanjyan for valuable advices, reviews and overall contribution to the creation of this article. Computations of matrix equations where completed using "Mathematica" software by Wolfram.

Conflict of Interest

There are no any conflicts of interest regarding this particular research.

Author Contributions

R.G. Petrosyan proposed the problem and performed the analytical research regarding the solution. M.A. Davtyan carried out theoretical calculations and wrote the text.

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