Armenian Journal of Mathematics
Volume 10, Number 10, 2018, 1–15

Subnexuses Based on $\mathcal{N}$-structures

(In memory of Dr. Hossein Hedayati)

M. Norouzi, A. Asadi and Y. B. Jun

Abstract. The notion of a subnexus based on $\mathcal{N}$-function (briefly, $\mathcal{N}$-subnexus) is introduced, and related properties are investigated. Also, the notions of $\mathcal{N}$-subnexus of type (α, β), where (α, β) is $(\in, \in)$, $(\in, q)$, $(\in, \in \vee q)$, $(q, \in)$, (q, q), $(q, \in \vee q)$, $(\overline{\in}, \overline{\in})$ and $(\overline{\in}, \overline{\in} \vee \overline{q})$, are introduced, and their basic properties are investigated. Conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$ are given, and characterizations of $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ and $(\overline{\in}, \overline{\in} \vee \overline{q})$ are provided. Homomorphic image and preimage of $\mathcal{N}$-subnexus are discussed.

Key Words: Nexus, subnexus, $\mathcal{N}$-structure, q-support, $\in \vee q$-support
Mathematics Subject Classification 2010: 08A72, 03E20, 03G15

Introduction

M. Bolourian in [4] defined nexus and studied properties of this structure algebra such as subnexuses, cyclic nexuses and homomorphism of nexuses. D. Afkhami et al. [1] defined the notion of fraction over a nexus and studied its basic properties. Moreover D. Afkhami et al. [2] defined soft nexuses over a nexus and studied the prime and maximal soft subnexuses over a nexus.

"The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [11] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in $BCK/BCI/BCH$-algebras" (see [11], [12], [13], [14]).

The notion of $(\in, \in \vee q)$-substructure, based on the concepts of belongingness and quasi-coincidence for a (fuzzy) point of a (fuzzy) subset, was defined and studied on many algebraic structures which some of them can be seen in [3], [5], [7], [8], [9] and [10]. Now, in this paper, we introduce the notion of a subnexus based on $\mathcal{N}$-function (briefly, $\mathcal{N}$-subnexus), and investigate related properties. We discuss characterization of $\mathcal{N}$-subnexus. We also introduce the notion of $\mathcal{N}$-subnexus of type (α, β) with

$$(\alpha, \beta) \in \{(\in, \in), (\in, q), (\in, \in \vee q), (q, \in), (q, q), (q, \in \vee q)\},$$

and investigate their basic properties. We provide conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$. We give characterizations of $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. We consider a condition for an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ to be an $\mathcal{N}$-subnexus of type $(\in, \in)$. We discuss homomorphic image and preimage of $\mathcal{N}$-subnexus. We also introduce the notions of $\mathcal{N}$-subnexus of types $(\overline{\in}, \overline{\in})$ and $(\overline{\in}, \overline{\in} \vee \overline{q})$, and establish characterizations of $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$.

1 Preliminaries

In this section we give some definitions and results which we need to develop our paper. They have been brought of [2, 6, 16], in connection with nexuses, and [12, 14] in connection with $\mathcal{N}$-structures.

An *address* is a sequence of $N^* = \mathbb{N} \cup \{0\}$ such that $a_k = 0$ implies that $a_i = 0$ for all $i \geq k$. The sequence of zero is called the *empty address* and denoted by ( ). In other word, every nonempty address is of the form $(a_1, a_2, \ldots, a_n, 0, 0, \ldots)$ where $n \in \mathbb{N}$, and it is denoted by $(a_1, a_2, \ldots, a_n)$. A set X of addresses is called a *nexus* if

(1) $(a_1, a_2, \ldots, a_n) \in X$ implies that $(a_1, \ldots, a_{n-1}, t) \in X$ for all $0 \leq t \leq a_n$.

(2) $(a_i)_{i=1}^{\infty} \in X$ implies that $(a_1, a_2, \ldots, a_n) \in X$ for all $n \in \mathbb{N}$.

Example 1 *A set $X = \{(), (1), (2), (3), (1, 1), (1, 2), (3, 1), (3, 2)\}$ is a nexus. But, $X' = \{(), (1), (2), (2, 2)\}$ is not a nexus since $(2, 2)$ is an element of X' but $(2, 1)$ is not in X'.*

Let X be a nexus and $w \in X$. The *level* of w, denoted by $l(w)$, is said to be: 0 if $w = ()$, n if $w = (a_1, a_2, \ldots, a_n)$ for some $a_n \in \mathbb{N}$, and ∞ if w is an infinite sequence of $\mathbb{N}$.

Definition 1 *Let $v = (a_i)$ and $w = (b_i)$ be addresses where $a_i, b_i \in \mathbb{N}$. Then $v \leq w$ if $l(v) = 0$ or one of the following cases is satisfied:*

(i) If $l(v) = 1$, i.e., $v = (a_1)$ for $a_1 \in \mathbb{N}$, then $l(w) \geq 1$ and $a_1 \leq b_1$.

(ii) If $1 < l(v) < \infty$, then $l(v) \leq l(w)$ and $a_{l(v)} \leq b_{l(v)}$ and for every $1 \leq i < l(v)$ we have, $a_i = b_i$.

(iii) If $l(v) = \infty$, then $v = w$.

Definition 2 *A nonempty subset S of a nexus X is called a subnexus of X if S itself is a nexus. The set of all subnexuses of X is denoted by $SUB(X)$.*

Note that a subset S of a nexus X is a subnexus of X if and only if it satisfies:

$$(\forall v, w \in X)(v \leq w,\ w \in S\ \Rightarrow\ v \in S). \tag{1}$$

Example 2 *Consider a nexus*

$$X = \{(), (1), (2), (3), (1,1), (2,1), (3,1), (3,1,1), (3,1,2)\}.$$

Then $X_1 = \{(), (1), (2), (3), (2,1)\}$, $X_2 = \{(), (1), (2), (1,1), (2,1)\}$ and $X_3 = \{(), (1), (2), (3), (3,1)\}$ are subnexuses of X.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Let $F(X, [-1, 0])$ be the set of all functions from the set X to $[-1, 0]$ (for briefly every element of $F(X, [-1, 0])$ is said to be $\mathcal{N}$-function on X). An $\mathcal{N}$-structure is a pair (X, f) of X and an $\mathcal{N}$-function f on X. For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0)$, the set $C(f; \alpha) = \{x \in X \mid f(x) \leq \alpha\}$ is called the *closed support* of (X, f) related to α, and the set $O(f; \alpha) = \{x \in X \mid f(x) < \alpha\}$ is said to be the *open support* of (X, f) related to α.

Let $\alpha \in [-1, 0)$ and (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(y) = \begin{cases} 0 & \text{if } y \neq x\ , \\ \alpha & \text{if } y = x. \end{cases}$$

In this case, f is denoted by x_α, and (X, x_α) is said to be a *point $\mathcal{N}$-structure with support x and value α*. For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_α) is an *$\mathcal{N}_\in$-subset* (resp. *$\mathcal{N}_q$-subset*) of (X, g) if $g(x) \leq \alpha$ (resp. $g(x) + \alpha + 1 < 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_\in$-subset or an $\mathcal{N}_q$-subset of (X, g), then we say (X, x_α) is an *$\mathcal{N}_{\in\vee q}$-subset* of (X, g).

2 Subnexus based on $\mathcal{N}$-function

In what follows, let X be a nexus unless otherwise specified.

Definition 3 *By a subnexus of X based on $\mathcal{N}$-function f (briefly, $\mathcal{N}$-subnexus of X), we mean an $\mathcal{N}$-structure (X, f) in which $w \leq v$ implies that $f(w) \leq f(v)$, for all $v, w \in X$.*

Example 3 *Let (X, f) be an $\mathcal{N}$-structure in which f is defined as $f(()) = -1$ and $f(w_n) = \frac{-1}{l(w_n)}$ for any $w_n := (a_1, a_2, \ldots, a_n) \in X$ with $w_n \neq ()$. Then (X, f) is an $\mathcal{N}$-subnexus of X.*

Example 4 *Let (X, f) be an $\mathcal{N}$-structure in which*

$$X = \{(), (1), (2), (2, 1), (2, 2)\},$$

is a nexus and f is defined by $f(()) = -0.9$, $f((1)) = -0.7$, $f((2)) = -0.6$, $f((2, 1)) = -0.4$ and $f((2, 2)) = -0.2$. Then (X, f) is an $\mathcal{N}$-subnexus of X.

Proposition 1 *If (X, f) is an $\mathcal{N}$-subnexus of X, then $f(()) \leq f(v)$ for all $v \in X$.*

Proof. Since $l(()) = 0$, we have $() \leq v$ for every $v \in X$. It follows that $f(()) \leq f(v)$ for all $v \in X$. □

Theorem 1 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of X if and only if for ever $\alpha \in [-1, 0)$, the non-empty closed support of (X, f) related to α is a subnexus of X.*

Proof. Assume that (X, f) is an $\mathcal{N}$-subnexus of X and $\alpha \in [-1, 0)$ be such that $C(f; \alpha) \neq \emptyset$. Let $w \in C(f; \alpha)$ and $v \leq w$. Then $f(w) \leq \alpha$ and $f(v) \leq f(w)$. Hence $f(v) \leq \alpha$ and so $v \in C(f; \alpha)$. Therefore $C(f; \alpha)$ is a subnexus of X. Conversely, let $C(f, \alpha)$ be a subnexus of X for any $\alpha \in [-1, 0)$ and $v \leq w$ for any $v, w \in X$. Taking $f(w) = \alpha$ implies that $w \in C(f; \alpha)$ and so $v \in C(f; \alpha)$. Thus $f(v) \leq \alpha = f(w)$ and so (X, f) is an $\mathcal{N}$-subnexus of X. □

Theorem 2 *For any $\mathcal{N}$-structure (X, f), the following are equivalent:*

(1) *(X, f) is an $\mathcal{N}$-subnexus of X.*

(2) *For any $v, w \in X$ with $v \leq w$ and $\alpha \in [-1, 0)$, if the point $\mathcal{N}$-structure (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f), then the point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f).*

Proof. Assume that (X, f) is an $\mathcal{N}$-subnexus of X. For any $v, w \in X$ with $v \leq w$, let $\alpha \in [-1, 0)$ be such that (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f). Then $f(w) \leq \alpha$, and so $f(v) \leq \alpha$. Hence (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f). Conversely suppose that (2) is valid and let $v, w \in X$ be such that $v \leq w$. Note that $(X, w_{f(w)})$ is an $\mathcal{N}_\in$-subset of (X, f). It follows from (2) that the point $\mathcal{N}$-structure $(X, v_{f(w)})$ is an $\mathcal{N}_\in$-subset of (X, f). Thus $f(v) \leq f(w)$, and consequently (X, f) is an $\mathcal{N}$-subnexus of X. □

Definition 4 *An $\mathcal{N}$-structure (X, f) is said to be an $\mathcal{N}$-subnexus of type*

(i) $(\in, \in)$ *(resp.,* $(\in, q)$ *and* $(\in, \in \vee q)$*) if whenever the point* $\mathcal{N}$*-structure* (X, w_α) *is an* $\mathcal{N}_\in$*-subset of* (X, f) *then the point* $\mathcal{N}$*-structure* (X, v_α) *is an* $\mathcal{N}_\in$*-subset (resp.,* $\mathcal{N}_q$*-subset and* $\mathcal{N}_{\in \vee q}$*-subset) of* (X, f) *for all* $v, w \in X$ *with* $v \leq w$ *and* $\alpha \in [-1, 0)$*.*

(ii) $(q, \in)$ *(resp.,* (q, q) *and* $(q, \in \vee q)$*) if whenever the point* $\mathcal{N}$*-structure* (X, w_α) *is an* $\mathcal{N}_q$*-subset of* (X, f) *then the point* $\mathcal{N}$*-structure* (X, v_α) *is an* $\mathcal{N}_\in$*-subset (resp.,* $\mathcal{N}_q$*-subset and* $\mathcal{N}_{\in \vee q}$*-subset) of* (X, f) *for all* $v, w \in X$ *with* $v \leq w$ *and* $\alpha \in [-1, 0)$*.*

It is easy to show that the notion of $\mathcal{N}$-subnexus of type $(\in, \in)$ is equivalent to the notion of $\mathcal{N}$-subnexus.

Example 5 *(1) It is easy to see that* $\mathcal{N}$*-structure* (X, f) *defined in Example 3, is an* $\mathcal{N}$*-subnexus of type* $(\in, \in)$*.*

(2) Let (X, f) *be an* $\mathcal{N}$*-structure in which* $X = \{(), (1), (2,1), (2,2)\}$ *is a nexus and* f *is defined by* $f = \begin{pmatrix} () & (1) & (2,1) & (2,2) \\ -1 & -1 & -1 & -0.93 \end{pmatrix}$*. It is routine to verify that* (X, f) *is an* $\mathcal{N}$*-subnexus of type* $(\in, q)$ *and* $(q, \in)$*.*

Example 6 *Let* (X, g) *be an* $\mathcal{N}$*-structure in which*

$$X = \{(), (1), (2), (1,1), (1,2)\},$$

is a nexus and g *is defined by* $g = \begin{pmatrix} () & (1) & (2) & (1,1) & (1,2) \\ -0.9 & -0.8 & -0.7 & -0.6 & -0.5 \end{pmatrix}$*. Then* (X, g) *is an* $\mathcal{N}$*-subnexus of type* $(q, \in)$*,* (q, q) *and* $(q, \in \vee q)$*.*

Example 7 *Consider an* $\mathcal{N}$*-structure* (X, h) *in which*

$$X = \{(), (1), (1,1), (1,2), (1,3)\},$$

is a nexus and h *is defined by* $h = \begin{pmatrix} () & (1) & (1,1) & (1,2) & (1,3) \\ -0.1 & -0.2 & -0.3 & -0.4 & -0.45 \end{pmatrix}$*. Then* (X, h) *is not an* $\mathcal{N}$*-subnexus of type* $(q, \in)$*,* (q, q) *and* $(q, \in \vee q)$*. For example* $() \leq (1)$ *and the point* $\mathcal{N}$*-structure* $(X, (1)_{-0.81})$ *is an* $\mathcal{N}_q$*-subset of* (X, h) *since* $h((1)) - 0.81 + 1 = -0.2 - 0.81 + 1 = -0.01 < 0$*. But the point* $\mathcal{N}$*-structure* $(X, ()_{-0.81})$ *is neither an* $\mathcal{N}_q$*-subset nor an* $\mathcal{N}_\in$*-subset of* (X, h)*, because of* $h(()) - 0.81 + 1 = -0.1 - 0.81 + 1 = 0.09 > 0$ *and* $h(()) = -0.1 > -0.81$*.*

Example 8 *Let* (X, g) *be an* $\mathcal{N}$*-structure in which*

$$X = \{(), (1), (2), (2,1), (2,2)\},$$

is a nexus and g *is defined by* $g = \begin{pmatrix} () & (1) & (2) & (2,1) & (2,2) \\ -0.63 & -0.67 & -0.65 & -0.83 & -0.72 \end{pmatrix}$*. It is easy to see that* (X, g) *is an* $\mathcal{N}$*-subnexus of type* $(\in, \in \vee q)$*.*

Theorem 3 *If (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$, $(\in, q)$ or $(q, \in)$, then the open support of (X, f) related to 0 is a subnexus of X.*

Proof. Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in)$. If $f(x) = 0$ for all $x \in X$, then $O(f; 0) = \emptyset$ which is a subnexus of X. Assume that f is nonzero and let $v, w \in X$ be such that $v \leq w$ and $w \in O(f; 0)$. Then $f(v) \leq f(w) < 0$, and so $v \in O(f; 0)$. Thus $O(f; 0)$ is a subnexus of X. Secondly, assume that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, q)$. Let $v, w \in X$ be such that $v \leq w$ and $w \in O(f; 0)$. Note that $(X, w_{f(w)})$ is an $\mathcal{N}_{\in}$-subset of (X, f). If $f(v) = 0$, then $f(v) + f(w) + 1 = f(w) + 1 \geq 0$. Thus $(X, v_{f(w)})$ is not an $\mathcal{N}_q$-subset of (X, f), a contradiction. Hence $f(v) < 0$, that is, $v \in O(f; 0)$. Hence $O(f; 0)$ is a subnexus of X. Finally, suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(q, \in)$. Let $v, w \in X$ be such that $v \leq w$ and $w \in O(f; 0)$. Then (X, w_{-1}) is an $\mathcal{N}_q$-subset of (X, f). If $f(v) = 0$, then (X, v_{-1}) is not an $\mathcal{N}_{\in}$-subset of (X, f). This is a contradiction, and so $f(v) < 0$, i.e., $v \in O(f; 0)$. Therefore $O(f; 0)$ is a subnexus of X. □

We provide conditions for an $\mathcal{N}$-structure to be an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$.

Theorem 4 *Let S be a subnexus of X and let (X, f) be an $\mathcal{N}$-structure such that*

(1) $(\forall x \in X)(x \in S \Rightarrow f(x) \leq -0.5)$,

(2) $(\forall x \in X)(x \notin S \Rightarrow f(x) = 0)$.

Then (X, f) is an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$.

Proof. Let $v, w \in X$ with $v \leq w$ and $\alpha \in [-1, 0)$ be such that the point $\mathcal{N}$-structure (X, w_{α}) is an $\mathcal{N}_q$-subset of (X, f). Then $f(w) + \alpha + 1 < 0$. Thus $v \in S$ because if $v \notin S$, then $w \notin S$. Hence $f(w) = 0$ and so $f(w) + \alpha + 1 = \alpha + 1 < 0$, that is, $\alpha < -1$, this is a contradiction. Therefore $f(v) \leq -0.5$. If $\alpha < -0.5$, then $f(v) + \alpha + 1 < -0.5 - 0.5 + 1 = 0$ and thus the point $\mathcal{N}$-structure (X, v_{α}) is an $\mathcal{N}_q$-subset of (X, f). If $\alpha \geq -0.5$, then $f(v) \leq -0.5 \leq \alpha$ and so the point $\mathcal{N}$-structure (X, v_{α}) is an $\mathcal{N}_{\in}$-subset of (X, f). Thus the point $\mathcal{N}$-structure (X, v_{α}) is an $\mathcal{N}_{\in \vee q}$-subset of (X, f), and therefore (X, f) is an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$. □

Theorem 5 *Let (X, f) be an $\mathcal{N}$-subnexus of type $(q, \in \vee q)$. If f is not constant on the open support of (X, f) related to 0 and $f(()) \geq f(x)$ for all $x \in X$, then there exists $y \in X$ such that $f(y) \leq -0.5$. In particular, $f(()) \leq -0.5$.*

Proof. Assume that $f(x) > -0.5$ for all $x \in X$. Since f is not constant on $O(f;0)$, there exists $y \in O(f;0)$ such that $\alpha_y = f(y) \neq f(()) = \alpha_0$. Then $\alpha_0 > \alpha_y$. Choose $\beta < -0.5$ such that $\alpha_0 + \beta + 1 > 0 > \alpha_y + \beta + 1$. Then the point $\mathcal{N}$-structure (X, y_β) is an $\mathcal{N}_q$-subset of (X, f). Since $() \leq y$, it follows that $(X, ()_\beta)$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f). But $f(()) > -0.5 > \beta$ implies that the point $\mathcal{N}$-structure $(X, ()_\beta)$ is not an $\mathcal{N}_\in$-subset of (X, f). Also $f(()) + \beta + 1 = \alpha_0 + \beta + 1 > 0$ implies that $(X, ()_\beta)$ is not an $\mathcal{N}_q$-subset of (X, f). This is a contradiction, and thus $f(y) \leq -0.5$ for some $y \in X$. We now prove that $f(()) \leq -0.5$. Assume that $\alpha_0 := f(()) > -0.5$. Note that there exists $y \in X$ such that $\alpha_y := f(y) \leq -0.5$ and so $\alpha_y < \alpha_0$. Choose $\alpha_1 < \alpha_0$ such that $\alpha_y + \alpha_1 + 1 < 0 < \alpha_0 + \alpha_1 + 1$. Then $f(y) + \alpha_1 + 1 = \alpha_y + \alpha_1 + 1 < 0$, and thus the point $\mathcal{N}$-structure (X, y_{α_1}) is an $\mathcal{N}_q$-subset of (X, f). Since $() \leq y$, we know that $(X, ()_{\alpha_1})$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f). But $f(()) + \alpha_1 + 1 = \alpha_0 + \alpha_1 + 1 > 0$ and also $f(()) = \alpha_0 > \alpha_1$ which is a contradiction. Therefore $f(()) \leq -0.5$. □

Theorem 6 *If (X, f) is an $\mathcal{N}$-subnexus of type (q, q) such that $f(()) \geq f(x)$ for all $x \in X$, then f is constant on the open support of (X, f) related to* 0.

Proof. Assume that f is not constant on the open support of (X, f) related to 0. Then there exists $x \in O(f;0)$ such that $\alpha_x = f(x) \neq f(()) = \alpha_0$. Then $\alpha_0 > \alpha_x$. and so $f(x) + (-1 - \alpha_0) + 1 = \alpha_x - \alpha_0 < 0$. Hence $(X, x_{-1-\alpha_0})$ is an $\mathcal{N}_q$-subset of (X, f). Note that $() \leq x$ and $f(()) + (-1 - \alpha_0) + 1 = \alpha_0 - \alpha_0 = 0$, which implies that $(X, ()_{-1-\alpha_0})$ is not an $\mathcal{N}_q$-subset of (X, f). This is impossible, and therefore f is constant on the open support of (X, f) related to 0. □

In the following, we give some characterizations for an $\mathcal{N}$-subnexus of type $(\in, \in\vee q)$.

Theorem 7 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in\vee q)$ if and only if the following assertion is valid.*

$$(\forall v, w \in X)\left(v \leq w \ \Rightarrow\ f(v) \leq \bigvee\{f(w), -0.5\}\right). \tag{2}$$

Proof. Suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in\vee q)$. For any $v, w \in X$ such that $v \leq w$, assume that $f(w) > -0.5$. If $f(v) > f(w)$, then there exists $\beta \in [-1, 0)$ such that $f(v) > \beta \geq f(w)$. Thus the point $\mathcal{N}$-structure (X, w_β) is an $\mathcal{N}_\in$-subset of (X, f), but the point $\mathcal{N}$-structure (X, v_β) is not an $\mathcal{N}_\in$-subset of (X, f). Also $f(v) + \beta + 1 > 2\beta + 1 \geq 2f(w) + 1 > 0$ and so (X, v_β) is not an $\mathcal{N}_q$-subset of (X, f). Therefore (X, v_β) is not an $\mathcal{N}_{\in\vee q}$-subset of (X, f), which is a contradiction. Hence $f(v) \leq f(w)$ whenever $f(w) > -0.5$. Now, suppose that $f(w) \leq -0.5$. Then the point $\mathcal{N}$-structure $(X, w_{-0.5})$ is an $\mathcal{N}_{\in}$-subset of (X, f) and so $(X, v_{-0.5})$ is an $\mathcal{N}_{\in\vee q}$-subset of (X, f) by hypothesis. If $(X, v_{-0.5})$ is an $\mathcal{N}_{\in}$-subset of (X, f) then $f(v) \leq -0.5$ and so $f(v) \leq \bigvee\{f(w), -0.5\}$. If $(X, v_{-0.5})$ is an $\mathcal{N}_q$-subset of (X, f), then $f(v) - 0.5 + 1 < 0$, that is, $f(v) < -0.5$. Consequently $f(v) \leq \bigvee\{f(w), -0.5\}$. Conversely, assume that (2) is valid. Let $v, w \in X$ and $\beta \in [-1, 0)$ be such that $v \leq w$ and the point $\mathcal{N}$-structure (X, w_β) is an $\mathcal{N}_{\in}$-subset of (X, f). If $f(v) \leq \beta$, then the point $\mathcal{N}$-structure (X, v_β) is an $\mathcal{N}_{\in}$-subset of (X, f). Suppose that $f(v) > \beta$. Then $f(w) \leq \beta < f(v) \leq \bigvee\{f(w), -0.5\}$, and therefore $\bigvee\{f(w), -0.5\} = -0.5$. It follows that

$$f(v) + \beta + 1 < 2f(v) + 1 \leq 2\left(\bigvee\{f(w), -0.5\}\right) + 1 = 0.$$

Thus (X, v_β) is an $\mathcal{N}_q$-subset of (X, f). Consequently (X, v_β) is an $\mathcal{N}_{\in\vee q}$-subset of (X, f) and thus (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. □

Theorem 8 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ if and only if for every $\alpha \in [-0.5, 0]$ the nonempty closed support of (X, f) related to α is a subnexus of X.*

Proof. Assume that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ and let $\alpha \in [-0.5, 0]$ such that $C(f; \alpha) \neq \emptyset$. Let $v \leq w$ and $w \in C(f; \alpha)$. Then $f(v) \leq \bigvee\{f(w), -0.5\}$ by Theorem 3.16. If $\bigvee\{f(w), -0.5\} = f(w)$, then $f(v) \leq f(w) \leq \alpha$ and thus $v \in C(f; \alpha)$. Also, if $\bigvee\{f(w), -0.5\} = -0.5$, then $f(v) \leq -0.5 \leq \alpha$, and thus $v \in C(f; \alpha)$. Hence $C(f; \alpha)$ is a subnexus of X. Conversely, let (X, f) be an $\mathcal{N}$-structure such that the nonempty closed support of (X, f) related to α is a subnexus of X for all $\alpha \in [-0.5, 0]$. If there exist $v, w \in X$ such that $v \leq w$ and $f(v) > \bigvee\{f(w), -0.5\}$, then we can take $\beta \in [-1, 0]$ such that $f(v) > \beta \geq \bigvee\{f(w), -0.5\}$. Thus $w \in C(f; \beta)$ and $\beta \geq -0.5$. Since $C(f; \beta)$ is a subnexus of X, we have $v \in C(f; \beta)$. Hence $f(v) \leq \beta$, a contradiction. Therefore $f(v) \leq \bigvee\{f(w), -0.5\}$ for all $v, w \in X$. It follows from Theorem 3.16 that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. □

Theorem 9 *Let S be a subnexus of X. For any $\alpha \in (-0.5, 0)$, there exists an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ for which S is represented by the closed support of (X, f) related to α.*

Proof. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(x) = \begin{cases} \alpha & \text{if } x \in S\ , \\ 0 & \text{if } x \notin S, \end{cases}$$

for all $x \in X$ where $\alpha \in (-0.5, 0)$. Assume that $f(v) > \bigvee\{f(w), -0.5\}$ for some $v, w \in X$ with $v \leq w$. Since the cardinality of the image of f is two, we have $f(v) = 0$ and $\bigvee\{f(w), -0.5\} = \alpha$. Since $\alpha > -0.5$, it follows that $f(w) = \alpha$. So $w \in S$. Since S is a subnexus of X, we obtain $v \in S$ and so $f(v) = \alpha < 0$ which is a contradiction. Therefore $f(v) \leq \bigvee\{f(w), -0.5\}$ for all $v, w \in X$. Hence (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ by Theorem 3.16. Obviously, S is represented by the closed support of (X, f) related to α. $\Box$

Note that every $\mathcal{N}$-subnexus of type $(\in, \in)$ is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. But the converse is not true in general as seen in the following example.

Example 9 *Let (X, f) be an $\mathcal{N}$-structure in which*

$$X = \{(), (1), (2), (1,1), (1,2), (1,3), (1,3,1), (1,3,2)\}$$

is a nexus and f is defined as follows:

$$h = \begin{pmatrix} () & (1) & (2) & (1,1) & (1,2) & (1,3) & (1,3,1) & (1,3,2) \\ -1 & -0.9 & -0.93 & -0.95 & -0.94 & -0.96 & -0.97 & -0.99 \end{pmatrix}$$

Then (X, h) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. But it is not an $\mathcal{N}$-subnexus of type $(\in, \in)$. In fact, set $\alpha = -0.92$. Note that $(1) \leq (2)$ and $h((2)) = -0.93 \leq \alpha$ but $h((1)) = -0.9 \not\leq \alpha$.

Now, we give a condition for an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ to be an $\mathcal{N}$-subnexus of type $(\in, \in)$.

Theorem 10 *Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ such that $f(x) > -0.5$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$.*

Proof. Let $v, w \in X$ such that $v \leq w$ and (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) for $\alpha \in [-1, 0)$. Then $f(w) \leq \alpha$. It follows from Theorem 3.16 and the hypothesis that $f(v) \leq \bigvee\{f(w), -0.5\} = f(w) \leq \alpha$. Thus (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$. $\Box$

Lemma 1 *If $\{(X, f_i) \mid i \in I\}$ is the class of $\mathcal{N}$-subnexuses, then its union $\left(X, \bigcup_{i\in I} f_i\right)$ and its intersection $\left(X, \bigcap_{i\in I} f_i\right)$ are also $\mathcal{N}$-subnexuses where $\bigcup_{i\in I} f_i$ and $\bigcap_{i\in I} f_i$ are given as follows:*

$$\bigcup_{i\in I} f_i : X \to [-1, 0],\ x \mapsto \bigvee_{i\in I} f_i(x), \quad \text{and} \quad \bigcap_{i\in I} f_i : X \to [-1, 0],\ x \mapsto \bigwedge_{i\in I} f_i(x). \tag{3}$$

Proof. Straightforward. $\Box$

Theorem 11 *If* $\{(X, f_i) \mid i \in I\}$ *is the class of* $\mathcal{N}$*-subnexuses of type* $(\in, \in \vee q)$*, then* $\left(X, \bigcup_{i\in I} f_i\right)$ *and* $\left(X, \bigcap_{i\in I} f_i\right)$ *are also* $\mathcal{N}$*-subnexuses of type* $(\in, \in \vee q)$*.*

Proof. Let $v, w \in X$ be such that $v \leq w$ and let $\alpha \in [-1, 0)$. Assume that (X, f_i) is an $\mathcal{N}$-subnexuses of type $(\in, \in \vee q)$ for all $i \in I$. Then $\left(X, \bigcup_{i\in I} f_i\right)$ is an $\mathcal{N}$-subnexus by Lemma 3.21 and $f_i(v) \leq \bigvee\{f_i(w), -0.5\}$ for all $i \in I$ by Theorem 3.16. Hence

$$\begin{aligned}\left(\bigcup_{i\in I} f_i\right)(v) &= \bigvee_{i\in I} f_i(v) \leq \bigvee_{i\in I}\left(\bigvee\{f_i(w), -0.5\}\right) = \bigvee\left\{\bigvee_{i\in I} f_i(w), \bigvee_{i\in I} -0.5\right\} \\ &= \bigvee\left\{\left(\bigcup_{i\in I} f_i\right)(w), -0.5\right\},\end{aligned}$$

and therefore $\left(X, \bigcup_{i\in I} f_i\right)$ is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. Similarly, we can show that $\left(X, \bigcap_{i\in I} f_i\right)$ is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ since $([-1, 0], \vee, \wedge)$ is a distributive lattice. □

A mapping $\varphi : X \to Y$ of nexuses is called a *homomorphism* if $v \leq w$ implies $\varphi(v) \leq \varphi(w)$ for all $v, w \in X$.

Definition 5 *Let* $\varphi : X \to Y$ *be a mapping of nexuses. Given an* $\mathcal{N}$*-subnexus* (X, f)*, its image under* φ *is defined to be an* $\mathcal{N}$*-subnexus* $(Y, \varphi(f))$ *for which* $\varphi(f)$ *is defined by*

$$\varphi(f) : Y \to [-1, 0],\ y \mapsto \begin{cases} \bigvee_{x\in\varphi^{-1}(y)} f(x) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{if } \varphi^{-1}(y) = \emptyset. \end{cases} \tag{4}$$

Also the preimage of an $\mathcal{N}$*-subnexus* (Y, g) *under* φ *is defined to be an* $\mathcal{N}$*-subnexus* $(X, \varphi^{-1}(g))$ *where* $\varphi^{-1}(g) : X \to [-1, 0]$*, by* $x \mapsto g(\varphi(x))$*.*

Theorem 12 *Let* $\varphi : X \to Y$ *be an onto homomorphism of nexuses. If* (X, f) *is an* $\mathcal{N}$*-subnexus, then so is its image* $(Y, \varphi(f))$ *under* φ*.*

Proof. Let $v', w' \in Y$ be such that $v' \leq w'$. Then there exist $v, w \in X$ such that $v' = \varphi(v)$ and $w' = \varphi(w)$, that is, $v \in \varphi^{-1}(v')$ and $w \in \varphi^{-1}(w')$. Since φ is a homomorphism, we have $v \leq w$ and so $f(v) \leq f(w)$. It follows that $\varphi(f)(v') = \bigvee_{v\in\varphi^{-1}(v')} f(v) \leq \bigvee_{w\in\varphi^{-1}(w')} f(w) = \varphi(f)(w')$. Therefore $(Y, \varphi(f))$ is an $\mathcal{N}$-subnexus. □

Theorem 13 *Let $\varphi : X \to Y$ be a homomorphism of nexuses. If (Y, g) is an $\mathcal{N}$-subnexus, then so is its preimage $(X, \varphi^{-1}(g))$ under φ.*

Proof. Let $v, w \in X$ such that $v \leq w$. Since φ is a homomorphism, we have $\varphi(v) \leq \varphi(w)$. It follows that $g(\varphi(v)) \leq g(\varphi(w))$ because (Y, g) is $\mathcal{N}$-subnexus. Thus $\varphi^{-1}(g)(v) \leq \varphi^{-1}(g)(w)$, and therefore $(X, \varphi^{-1}(g))$ is an $\mathcal{N}$-subnexus. □

For any $\mathcal{N}$-structure (X, f) and $\alpha \in [-1, 0)$, the *q-support* and the $\in \vee q$-*support* of (X, f) related to α are defined as follows:

$$\mathcal{N}_q(f;\alpha) = \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_q\text{-subset of } (X, f)\}, \text{ and}$$
$$\mathcal{N}_{\in\vee q}(f;\alpha) = \{x \in X \mid (X, x_\alpha) \text{ is an } \mathcal{N}_{\in\vee q}\text{-subset of } (X, f)\}.$$

Note that the $\in \vee q$-support is the union of the closed support and the q-support, that is, $\mathcal{N}_{\in\vee q}(f;\alpha) = C(f;\alpha) \cup \mathcal{N}_q(f;\alpha)$.

Theorem 14 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$ if and only if the $\in \vee q$-support of (X, f) related to α is a subnexus of X for all $\alpha \in [-1, 0)$.*

Proof. Suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. Let $v, w \in X$ such that $v \leq w$ and $w \in \mathcal{N}_{\in\vee q}(f;\alpha)$ for $\alpha \in [-1, 0)$. Then (X, w_α) is an $\mathcal{N}_{\in\vee q}$-subset of (X, f). Thus $f(w) \leq \alpha$ or $f(w) + \alpha + 1 < 0$. If $f(w) \leq \alpha$, then (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f) and so (X, v_α) is an $\mathcal{N}_{\in\vee q}$-subset of (X, f) by hypothesis. Hence $v \in \mathcal{N}_{\in\vee q}(f;\alpha)$. Suppose that $f(w) + \alpha + 1 < 0$. If $\alpha \geq -0.5$, then $f(w) < -0.5$ which implies from (2) that $f(v) \leq \bigvee\{f(w), -0.5\} = -0.5 \leq \alpha$. Thus $v \in C(f;\alpha) \subseteq \mathcal{N}_{\in\vee q}(f;\alpha)$. If $\alpha < -0.5$, then $f(v) + \alpha + 1 < f(v) + 0.5 \leq \bigvee\{f(w), -0.5\} + 0.5 = 0$ whenever $\bigvee\{f(w), -0.5\} = -0.5$, and $f(v) + \alpha + 1 \leq \bigvee\{f(w), -0.5\} + \alpha + 1 = f(w) + \alpha + 1 < 0$ whenever $\bigvee\{f(w), -0.5\} = f(w)$. Hence $v \in \mathcal{N}_q(f;\alpha) \subseteq \mathcal{N}_{\in\vee q}(f;\alpha)$. Therefore $\mathcal{N}_{\in\vee q}(f;\alpha)$ is a subnexus of X for all $\alpha \in [-1, 0)$. Conversely, let (X, f) be an $\mathcal{N}$-structure for which $\in \vee q$-support of (X, f) related to α is a subnexus of X for all $\alpha \in [-1, 0)$. Assume that there exist $v, w \in X$ such that $v \leq w$ and $f(v) > \bigvee\{f(w), -0.5\}$. Then $f(v) > \beta \geq \bigvee\{f(w), -0.5\}$ for some $\beta \in [-0.5, 0)$. It follows that $w \in C(f;\beta) \subseteq \mathcal{N}_{\in\vee q}(f;\beta)$ but $v \notin C(f;\beta)$. Also $f(v) + \beta + 1 > 2\beta + 1 \geq 0$, that is, $v \notin \mathcal{N}_q(f;\beta)$. Thus $v \notin C(f;\beta) \cup \mathcal{N}_q(f;\beta) = \mathcal{N}_{\in\vee q}(f;\beta)$ which is a contradiction. Therefore $f(v) \leq \bigvee\{f(w), -0.5\}$ for all $v, w \in X$ with $v \leq w$. It follows from Theorem 3.16 that (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in \vee q)$. □

3 $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$

For any $\mathcal{N}$-structure (X, g), we say that a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_{\overline{\in}}$-*subset* (resp., $\mathcal{N}_{\overline{q}}$-*subset*) of (X, g) if $g(x) > \alpha$ (resp., $g(x) + \alpha + 1 \geq 0$). If a point $\mathcal{N}$-structure (X, x_α) is an $\mathcal{N}_{\overline{\in}}$-subset or an $\mathcal{N}_{\overline{q}}$-subset of (X, g), we say (X, x_α) is an $\mathcal{N}_{\overline{\in} \vee \overline{q}}$-*subset* of (X, g).

Definition 6 *An $\mathcal{N}$-structure (X, f) is called an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ (resp., $(\overline{\in}, \overline{\in})$) if whenever (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f) then (X, w_α) is an $\mathcal{N}_{\overline{\in} \vee \overline{q}}$-subset (resp., $\mathcal{N}_{\overline{\in}}$-subset) of (X, f) for all $v, w \in X$ with $v \leq w$ and $\alpha \in [-1, 0)$.*

Example 10 *Let (X, f) be an $\mathcal{N}$-structure in which*

$$X = \{(), (1), (2), (2, 1), (2, 2)\},$$

is a nexus and f is given as $f = \begin{pmatrix} () & (1) & (2) & (2,1) & (2,2) \\ -0.32 & -0.41 & -0.43 & -0.5 & -0.48 \end{pmatrix}$. *It is easy to see that (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$.*

Theorem 15 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$ if and only if it is an $\mathcal{N}$-subnexus of type $(\in, \in)$.*

Proof. Let (X, f) be an $\mathcal{N}$-subnexus of type $(\in, \in)$ and $v, w \in X$ with $v \leq w$ such that (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f) for $\alpha \in [-1, 0)$. Then $f(v) > \alpha$. If $f(w) \leq \alpha$, then (X, w_α) is an $\mathcal{N}_{\in}$-subset of (X, f). Hence (X, v_α) is an $\mathcal{N}_{\in}$-subset of (X, f), and so $f(v) \leq \alpha$ which is a contradiction. Thus $f(w) > \alpha$, that is, (X, w_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in})$. Conversely, we can prove it by the similar way. $\square$

Theorem 16 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ if and only if the following condition is valid:*

$$(\forall v, w \in X)(v \leq w \;\Rightarrow\; \bigwedge\{f(v), -0.5\} \leq f(w)). \tag{5}$$

Proof. Suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. If there exist $v, w \in X$ such that $v \leq w$ and $\bigwedge\{f(v), -0.5\} > f(w) = \alpha$, then $\alpha \in [-1, -0.5)$. It follows that (X, w_α) is an $\mathcal{N}_{\in}$-subset of (X, f) and (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Hence (X, w_α) is an $\mathcal{N}_{\overline{q}}$-subset of (X, f). Therefore $f(w) + \alpha + 1 \geq 0$, which implies that $2\alpha + 1 \geq 0$, that is, $\alpha \geq -0.5$. This is contradiction, and so $\bigwedge\{f(v), -0.5\} \leq f(w)$ for all $v, w \in X$ with $v \leq w$. Conversely, assume that an $\mathcal{N}$-structure (X, f) satisfies the condition (5). Let $v \leq w$ for $v, w \in X$ and $\alpha \in [-1, 0)$ such that a point $\mathcal{N}$-structure (X, v_α) is an $\mathcal{N}_{\overline{\in}}$-subset of (X, f). Then $f(v) > \alpha$. If $f(v) \leq f(w)$, then

$\alpha < f(w)$. Thus (X, w_α) is an $\mathcal{N}_{\overline{\in}\vee\overline{q}}$-subset of (X, f). If $f(v) > f(w)$, then $\bigwedge\{f(v), -0.5\} = -0.5$ by (5). Hence $-0.5 \leq f(w)$. Suppose that (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f). Then $\alpha \geq f(w) \geq -0.5$. It follows that $f(w)+\alpha+1 \geq 2f(w)+1 \geq 0$ and so that (X, w_α) is an $\mathcal{N}_{\overline{q}}$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. $\square$

Proposition 2 *If (X, f) is an $\mathcal{N}$-structure of type $(\overline{\in}, \overline{\in} \vee \overline{q})$, then $f(w) \geq f(())$ or $f(w) \geq -0.5$ for all $w \in X$.*

Proof. It is straightforward by (5). $\square$

Theorem 17 *An $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ if and only if the nonempty closed support of (X, f) related to α is a subnexus of X for every $\alpha \in [-1, -0.5)$.*

Proof. Suppose that (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. Let $v, w \in X$ with $v \leq w$ and $w \in C(f; \alpha)$ for $\alpha \in [-1, -0.5)$. Then $f(w) \leq \alpha$. It follows from (5) that $\bigwedge\{f(v), -0.5\} \leq f(w) \leq \alpha$. Since $\alpha < -0.5$, we have $f(v) \leq \alpha$ and so $v \in C(f; \alpha)$. Therefore $C(f; \alpha)$ is a subnexus of X. Conversely, let (X, f) be an $\mathcal{N}$-structure such that the nonempty closed support of (X, f) related to α is a subnexus of X for all $\alpha \in [-1, -0.5)$. Assume that there exist $v, w \in X$ such that $v \leq w$ and $\bigwedge\{f(v), -0.5\} > f(w)$. If we take $\beta := \frac{1}{2}(\bigwedge\{f(v), -0.5\}) + f(w)$, then $\beta \in [-1, -0.5)$ and $\bigwedge\{f(v), -0.5\}) > \beta \geq f(w)$. Thus $w \in C(f; \beta)$ but $v \notin C(f; \beta)$ which is a contradiction. Therefore $\bigwedge\{f(v), -0.5\}) \leq f(w)$ for all $v, w \in X$ with $v \leq w$. Using Theorem 4.4, we conclude that (X, f) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. $\square$

Obviously, every $\mathcal{N}$-subnexus of type $(\in, \in)$ is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$, but the converse is not true in general as seen in the following example.

Example 11 *Let $X = \{(), (1), (2), (1, 1), (1, 2), (1, 2, 1)\}$ be a nexus. Consider an $\mathcal{N}$-structure (X, h) in which h is defined by*

$$h = \begin{pmatrix} () & (1) & (2) & (1,1) & (1,2) & (1,2,1) \\ -0.4 & -0.45 & -0.33 & -0.3 & -0.34 & -0.2 \end{pmatrix}$$

It is routine to verify that (X, h) is an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$. But it is not an $\mathcal{N}$-subnexus of type $(\in, \in)$ since $() \leq (1)$ and $(X, (1)_{-0.42})$ is an $\mathcal{N}_\in$-subset of (X, h), but $(X, ()_{-0.42})$ is not an $\mathcal{N}_\in$-subset of (X, h).

We provide conditions for an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ to be an $\mathcal{N}$-subnexus of type $(\in, \in)$.

Theorem 18 *Let (X, f) be an $\mathcal{N}$-subnexus of type $(\overline{\in}, \overline{\in} \vee \overline{q})$ such that $f(x) \leq -0.5$ for all $x \in X$. Then (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$.*

Proof. Let $v, w \in X$ and $\alpha \in [-1, 0)$ be such that $v \leq w$ and (X, w_α) is an $\mathcal{N}_\in$-subset of (X, f). Then $f(w) \leq \alpha$. Since $f(x) \leq -0.5$ for all $x \in X$, it follows from (5), $f(v) = \bigwedge\{f(v), -0.5\}) \leq f(w) \leq \alpha$. Thus (X, v_α) is an $\mathcal{N}_\in$-subset of (X, f). Therefore (X, f) is an $\mathcal{N}$-subnexus of type $(\in, \in)$. □

References

[1] D. Afkhami, N. Ahmadkhah and A. Hasankhani, A fraction of nexuses, Int. J. Algebra, **1**8(2011) no. 5, pp. 883-896.
http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.407.8169

[2] D. Afkhami Taba, A. Hasankhani and M. Bolurian, Soft nexuses, Comput. Math. Appl., **6**4(2012) pp. 1812-1821.
https://www.sciencedirect.com/science/article/pii/S0898122112001733.

[3] S.K. Bhakat, P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy Sets and Systems, **8**0(1996) pp. 359-368.

[4] M. Bolourian, *Theory of plenices*, Ph. D. Thesis, University of Surrey, 2009.

[5] B. Davvaz, $(\in, \in \vee q)$-fuzzy subnearrings and ideals, Soft Comput., **1**0(2006) pp. 206-211.
https://link.springer.com/article/10.1007/s00500-005-0472-1.

[6] H. Hedayati and A. Asadi, Normal, maximal and product fuzzy subnexuses of nexuses, J. Intell. Fuzzy Syst., **2**6(2014) no. 3, pp. 1341-1348.
https://content.iospress.com/articles/journal-of-intelligent-and-fuzzy-systems/ifs820.

[7] Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc., **4**2(2005) no. 4, pp. 703-711.
http://bkms.kms.or.kr/

[8] Y. B. Jun, Generalizations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras, Comput. Math. Appl., **5**8(2009) pp. 1383-1390.
https://www.sciencedirect.com/science/article/pii/S0898122109004866.

[9] Y. B. Jun, Fuzzy subalgebras of type (α, β) in BCK/BCI-algebras, KYUNGPOOK Math. J., **4**7(2007) pp. 403-410.
http://kmj.knu.ac.kr/

[10] Y. B. Jun, S. Song and E. H. Roh, BCK-filters based on fuzzy points with thresholds, KYUNGPOOK Math. J., **5**1(2011) pp. 11-28. http://kmj.knu.ac.kr/

[11] Y. B. Jun, K. J. Lee and S. Z. Song, $\mathcal{N}$-ideals of BCK/BCI-algebras, J. Chungcheong Math. Soc., **2**2(2009) pp. 417-437.

[12] Y. B. Jun, M. S. Kang and C. H. Park, $\mathcal{N}$-subalgebras in BCK/BCI-algebras based on point $\mathcal{N}$-structures, Int. J. Math. Math. Sci., Volume 2010, Article ID 303412, 9 pages. https://www.hindawi.com/journals/ijmms/contents/

[13] Y. B. Jun, M. A. Ozturk and E. H. Roh, $\mathcal{N}$-structures applied to closed ideals in BCH-algebras, Int. J. Math. Math. Sci., vol. 2010, Article ID 943565, 9 pages, 2010. https://www.hindawi.com/journals/ijmms/contents/

[14] K. J. Lee, Y. B. Jun and X. Zhang, $\mathcal{N}$-subalgebras of type $(\in, \in \vee q)$ based on point $\mathcal{N}$-structures in BCK/BCI-algebras, Commun. Korean Math. Soc., **2**7(2012) no. 3, pp. 431-439. http://ckms.kms.or.kr/

[15] H. Nooshin and P. Disney, Formex configuration processin I, Int. J. Space Structures, **1**5(2000) no. 1, pp. 1-52. http://journals.sagepub.com/doi/pdf/10.1260/0266351001494955

[16] L. Torkzadeh and A. Hasankhani, Some results on prime and maximal subnexuses of a nexus, Proceeding of 20th Seminar on Algebra, Tarbiat Moallem University, Iran, 22-23 Apr. 2009, 222-224.

Morteza Norouzi
Department of Mathematics, Faculty of Basic Sciences,
University of Bojnord
P. O. Box 1339, Bojnord 94531, Iran.
m.norouzi65@yahoo.com, m.norouzi@ub.ac.ir

Ameneh Asadi
Department of Mathematics, Payame Noor University
Tehran, Iran.
asadi8232@yahoo.com

Young Bae Jun
Department of Mathematics Education, Gyeongsang National University
Jinju 52828, Korea.
skywine@gmail.com

Please, cite to this paper as published in
Armen. J. Math., V. **10**, N. 10(2018), pp. 1–15