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Stability, Boundedness, and Square Integrability of Solutions of Neutral Fourth-Order Differential Equations

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Abstract. The purpose of this paper is to establish a new result, which guarantees the asymptotic stability and boundedness of the zero solution and the square integrability of solutions and their derivatives to neutral type nonlinear differential equations of fourth order. We illustrate our results by an example at the end of the paper.

Key Words: Lyapunov Functional, Neutral Differential Equations of Fourth Order, Uniform Asymptotic Stability, Square Integrability
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Introduction

The investigation of qualitative behaviour of the solution of nonlinear delay differential equation of fourth order has received considerable attention and has been subject of many articles in the literature, for instance, Abou-El-Ela et al. [1], Bereketoglu [3], Chin [7], Ezeilo [9]-[12], Kang [18], Omeike [19], Rahmane and Fatmi and Remili [21], Rahmane and Remili [22], Remili and Rahmane [28, 29, 30], Sadek [31], Sinha [32], Tejumola and Tchegnani [33], Tunç [34], Vlček [35], Wu and Xiong [36]. For nonlinear differential equations of neutral type, there are few results of stability, boundedness, and square integrability of solutions.

In this article, we investigate some asymptotic properties of solutions of the fourth-order nonlinear neutral delay differential equation

$$\begin{aligned}\big(x(t)+\rho x\,(t-r)\big)'''' &+ a\,(t)\,x'''(t)+b\,(t)\,x''(t)+c\,(t)\,x'(t)\\ &+ d\,(t)\,h\,(x\,(t)) = p(t,x(t),x'(t),x''(t),x'''(t)),\end{aligned} \tag{1}$$

where ρ and r are positive constants to be determined later, $a(.), b(.), c(.), d(.)$, and $h(x)$ are continuous functions depending only on the arguments shown, and $h'(x)$ exists and is continuous. For the sake of convenience, we introduce the following notation

$$\begin{cases} X(t) = x(t) + \rho x(t-r), \\ Y(t) = x'(t) + \rho x'(t-r), \\ Z(t) = x''(t) + \rho x''(t-r), \\ W(t) = x'''(t) + \rho x'''(t-r). \end{cases}$$

By a solution of (1) we mean a continuous function $x : [t_x, \infty) \to \mathbb{R}$ such that $X(t) \in C^3([t_x, \infty), \mathbb{R})$ and which satisfies equation (1) on $[t_x, \infty)$.

Without further mention, we will assume throughout that every solution $x(t)$ of (1) under consideration here is continuable to the right and nontrivial, i.e, $x(t)$ is defined on some ray $[t_x, \infty)$. Moreover, we tacitly assume that (1) possesses such solutions.

The problem of interest here is to investigate conditions under which all solutions of (1) converge to zero and are square integrable. We shall use appropriate Lyapunov functions and impose suitable conditions on the function $h(x)$.

1 Assumptions and main results

We shall state here some assumptions which will be used on the functions that appeared in equation (1), and suppose that there are positive constants $a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, h_0, \delta$, and δ_0 such that the following conditions hold:

i) $0 < a_0 \le a(t) \le a_1,\ 0 < b_0 \le b(t) \le b_1,\ 0 < c_0 \le c(t) \le c_1,$
$0 < d_0 \le d(t) \le d_1,$ and $d'(t) \le 0$ for $t \ge 0$.

ii) $h(0) = 0\ ,\ \dfrac{h(x)}{x} \ge \delta > 0$ for $x \neq 0$.

iii) $h_0 - \dfrac{a_0\delta_0}{d_1} \le h'(x) \le \dfrac{h_0}{2}$ for $x \in \mathbb{R}$.

iv) $b_0 > \dfrac{c_1}{a_0} + \dfrac{a_1 h_0 d_1}{c_0} + \dfrac{\delta_0}{a_0} = \kappa.$

The following lemma will be useful in the proof of the next theorem.

Lemma 1 *[17] Let* $h(0) = 0,\ xh(x) > 0\ (x \neq 0)$ *and*

$$\delta(t) - h'(x) \ge 0\ (\delta(t) > 0).$$

Then,

$$2\delta(t)H(x) \ge h^2(x), \qquad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

The first main result in this paper establishes sufficient conditions under which all solutions of the fourth-order nonlinear differential equation (1) and their first, second, and third derivatives converge to zero as $t \to \infty$.

Theorem 1 *In addition to assumptions (i)–(iv), assume that there are positive constants* η_1 *and* η_2 *such that the following conditions are satisfied:*

H1) $\int_0^{+\infty} (|a'(t)| + |b'(t)| + |c'(t)| - d'(t))\, dt \leq \eta_1$;

H2) $|p(t,x,x',x'',x''')| \leq |e(t)|$ *and* $\int_0^{+\infty} |e(t)|\, dt < \eta_2$.

Then, there exists a finite positive constant K *such that every solution* $x(.)$ *of (1) and their derivatives* $x'(.), x''(.), x'''(.)$, *and* $X'''(.)$ *satisfy :*

1. $|x(t)| \leq \sqrt{K},\ |x'(t)| \leq \sqrt{K},\ |x''(t)| \leq \sqrt{K},\ |X'''(t)| \leq \sqrt{K}$, *for all* $t \geq 0$.

2. $\int_0^{\infty} \big(x^2(s) + x'^2(s) + x''^2(s) + x'''^2(s)\big) ds < \infty$,

provided that

$$\rho < \min\Big\{1,\ \frac{2\varepsilon}{\alpha h_0},\ \frac{2\varepsilon c_0}{\alpha c_1 + \alpha d_1 \lambda_0},\ 2\frac{b_0 - \kappa - \varepsilon(a_1 + c_1)}{\alpha b_1 + \beta + \alpha d_1 \lambda_0 + \alpha d_1}, \frac{2\varepsilon a_0}{\alpha(2a_1 + b_1 + c_1 + d_1) + 5 + \beta}\Big\}, \tag{2}$$

where

$$\alpha = \frac{1}{a_0} + \varepsilon\ ,\ \beta = \frac{d_1 h_0}{c_0} + \varepsilon \quad \text{and} \quad \varepsilon < \min\Big\{\frac{1}{a_0}\ ,\ \frac{d_1 h_0}{c_0}\ ,\ \frac{b_0 - \kappa}{a_1 + c_1}\Big\}. \tag{3}$$

Proof. We first will write equation (1) as the equivalent system

$$\begin{cases} x' = y, \qquad y' = z, \qquad z' = w, \\ W'(t) = -a(t)w - b(t)z - c(t)y - d(t)h(x) + p(t,x,y,z,w). \end{cases} \tag{4}$$

It easy to see from (4) that

$$\begin{cases} X'(t) = y(t) + \rho y(t-r) = Y(t) \\ X''(t) = z(t) + \rho z(t-r) = Z(t) \\ X'''(t) = w(t) + \rho w(t-r) = W(t). \end{cases}$$

Our main tool is the continuously differentiable function $U = U(t,x,y,z,w)$ defined by

$$U = G(t)V = e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds} V, \tag{5}$$

where $\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| - d'(t)$, the function $V = V(t,x,y,z,w)$ is defined by

$$\begin{aligned} 2V \ = \ & [a(t) - \beta + \alpha b(t)]z^2 + [2\beta a(t) + 2\alpha c(t)]yz + 2\beta yW + 2zW \\ & + 2d(t)h(x)y + 2\alpha d(t)h(x)Z + \big[\beta b(t) - \alpha h_0 d(t) + c(t)\big]y^2 \\ & + \alpha W^2 + \alpha\rho d(t)(z(t-r))^2 + 2\beta d(t)H(x) \\ & + \mu_1 \int_{t-r}^t z^2(s)\, ds + \mu_2 \int_{t-r}^t w^2(s)\, ds, \end{aligned}$$

and η is a positive constant to be determined later in the proof. By adding and subtracting some terms, we can rewrite $2V$ as

$$\begin{aligned} 2V &= V_1+V_2+V_3+V_4+a(t)\left[\frac{W}{a(t)}+z+\beta y\right]^2 \\ &+c(t)\left[\frac{d(t)h(x)}{c(t)}+y+\alpha z\right]^2+\frac{d^2(t)h^2(x)}{c(t)} \\ &+\mu_1\int_{t-r}^{t} z^2(s)\,ds+\mu_2\int_{t-r}^{t} w^2(s)\,ds, \end{aligned}$$

where

$$\begin{aligned} V_1 &= 2d(t)\int_0^x h(s)\left[\frac{d_1h_0}{c_0}-2\frac{d(t)}{c(t)}h'(s)\right]ds, \\ V_2 &= \left[\alpha b(t)-\beta-\alpha^2c(t)\right]z^2, \\ V_3 &= \left[\beta b(t)-\alpha h_0 d(t)-\beta^2a(t)\right]y^2+\left[\alpha-\frac{1}{a(t)}\right]W^2, \\ V_4 &= 2\varepsilon d(t)H(x)+2\alpha\rho d(t)h(x)z(t-r)+\alpha\rho d(t)\left(z(t-r)\right)^2. \end{aligned}$$

To prove that V is positive definite, it suffices to show that V_1, V_2, V_3, and V_4 are positives. Remark that the estimate (3) implies

$$\frac{1}{a_0}<\alpha<2\frac{1}{a_0}\quad\text{and}\quad\frac{d_1h_0}{c_0}<\beta<2\frac{d_1h_0}{c_0}. \tag{6}$$

Then, using conditions i) $\sim$ iv), and inequalities (3) and (6), we obtain

$$\begin{aligned} V_1 &\geq 2d(t)\int_0^x h(s)\frac{d_1}{c_0}\left[h_0-2h'(s)\right]ds \\ &\geq 4\frac{d_0d_1}{c_0}\int_0^x h(s)\left[\frac{h_0}{2}-h'(s)\right]ds\geq 0. \end{aligned}$$

Rearranging V_2, we obtain the estimate

$$\begin{aligned} V_2 &= \alpha\Big[b(t)-\beta a(t)-\alpha c(t)\Big]z^2+\beta\Big[\alpha a(t)-1\Big]z^2 \\ &\geq \alpha\Big[b(t)-\big(\frac{d_1h_0}{c_0}+\varepsilon\big)a(t)-\big(\frac{1}{a_0}+\varepsilon\big)c(t)\Big]z^2+\beta\Big[\frac{a(t)}{a_0}-1\Big]z^2 \\ &\geq \alpha\Big[b_0-\frac{a_1d_1h_0}{c_0}-\frac{c_1}{a_0}-\varepsilon\big(a_1+c_1\big)\Big]z^2 \\ &\geq \alpha\Big[b_0-\kappa-\varepsilon\left(a_1+c_1\right)\Big]z^2\geq 0. \end{aligned}$$

We also have,

$$\begin{aligned} V_3 &\geq \beta\Big(b_0 - \frac{\alpha}{\beta}h_0 d_1 - \beta a_1\Big)y^2 + \Big(\alpha - \frac{1}{a_0}\Big)W^2 \\ &\geq \beta\Big(b_0 - \frac{c_0}{a_0} - a_1\frac{d_1 h_0}{c_0} - \varepsilon(c_0 + a_1)\Big)y^2 + \varepsilon W^2 \\ &\geq \beta\big(b_0 - \kappa - \varepsilon(c_1 + a_1)\big)y^2 + \varepsilon W^2 \geq 0. \end{aligned}$$

From the estimate on ρ, we have

$$\begin{aligned} V_4 &= 2\varepsilon d\,(t)\int_0^x h(\xi)d\xi + \alpha\rho d\,(t)\left[(z\,(t-r) + h\,(x))^2 - h^2\,(x)\right] \\ &\geq 2\varepsilon d\,(t)\int_0^x h(\xi)d\xi - 2\alpha\rho d\,(t)\int_0^x h'(\xi)h(\xi)d\xi \\ &\geq 2d\,(t)\int_0^x \left(\varepsilon - \frac{\alpha\rho h_0}{2}\right)h(\xi)d\xi \\ &\geq 2d_0\left(\varepsilon - \frac{\alpha\rho h_0}{2}\right)H(x). \end{aligned}$$

Thus, there exists a positive number D_0 such that

$$2V \geq D_0\left(y^2 + z^2 + W^2 + H(x)\right).$$

By Lemma 1 and condition iii) we conclude that there exists a positive number D_1 such that

$$2V \geq D_1\left(x^2 + y^2 + z^2 + W^2\right); \tag{7}$$

thus, V is positive-definite. Then we can find positive-definite functions $U_1(\|\xi\|)$ and $U_2(\|\xi\|)$ such that $U_1(\|\xi\|) \leq V \leq U_2(\|\xi\|)$. By (5) and inequality (7), we get

$$U \geq D_2(x^2 + y^2 + z^2 + W^2), \tag{8}$$

where $D_2 = \dfrac{D_1}{2}e^{-\frac{\eta_1}{\eta}}$. Therefore, by conditions H1) and H2), we can find positive-definite functions $W_1(\|\xi\|)$ and $W_2(\|\xi\|)$ such that

$$W_1(\|\xi\|) \leq U \leq W_2(\|\xi\|).$$

Now we prove that $\dot{U}$ is a negative-definite function. Along any solution $(x(t), y(t), z(t), w(t))$ of system (4), we have

$$2\dot{V}_{(4)} = V_5 + V_6 + V_7 + V_8 + V_9 + 2(\beta y + z + \alpha W)p(t, x, y, z, w),$$

where

$V_5 = -2\left(\frac{d_1 h_0}{c_0} c\left(t\right) - d\left(t\right) h'\left(x\right)\right) y^2 - 2\alpha d\left(t\right)\big(h_0 - h'\left(x\right)\big) yz,$

$V_6 = -2\big(b\left(t\right) - \alpha c\left(t\right) - \beta a\left(t\right)\big) z^2,$

$V_7 = -2\big(\alpha a\left(t\right) - 1\big) w^2,$

$$\begin{aligned} V_8 \;=\; & -2\varepsilon c\left(t\right) y^2 - 2\alpha\rho a(t) w_t w - 2\alpha\rho b(t) z w_t - 2\alpha\rho c\left(t\right) y w_t \\ & +2\alpha\rho d\left(t\right) h'\left(x\right) y z_t + \mu_1 z^2 + \mu_2 w^2 - \mu_1 z_t^2 - \mu_2 w_t^2 \\ & +2\alpha\rho d\left(t\right) z_t w_t + 2\rho w w_t + 2\beta\rho z w_t, \end{aligned}$$

$$\begin{aligned} V_9 \;=\; & d'\left(t\right)\big[2\beta H\left(x\right) - \alpha h_0 y^2 + 2h\left(x\right) y + 2\alpha h\left(x\right) z\big] + c'\left(t\right)\big[y^2 + 2\alpha yz\big] \\ & +b'\left(t\right)\big[\alpha z^2 + \beta y^2\big] + a'\left(t\right)\big[z^2 + 2\beta yz\big] + \alpha\rho d'\left(t\right)\left[z\left(t-r\right) + h(x)\right]^2 \\ & -\alpha\rho d'\left(t\right) h^2(x). \end{aligned}$$

Again, using conditions i), iii), iv), and inequalities (3) and (6), we get

$$\begin{aligned} V_5 &\leq -2\left[d\left(t\right) h_0 - d\left(t\right) h'\left(x\right)\right] y^2 - 2\alpha d\left(t\right)\left[h_0 - h'\left(x\right)\right] yz \\ &\leq -2d\left(t\right)\left[h_0 - h'\left(x\right)\right]\left[\left(y + \frac{\alpha}{2} z\right)^2 - \left(\frac{\alpha}{2} z\right)^2\right] \\ &\leq \frac{\alpha^2}{2} d\left(t\right)\left[h_0 - h'\left(x\right)\right] z^2. \end{aligned}$$

Therefore,

$$\begin{aligned} V_5 + V_6 &\leq -2\left[b\left(t\right) - \alpha c\left(t\right) - \beta a\left(t\right) - \frac{\alpha^2}{4} d\left(t\right)\left[h_0 - h'\left(x\right)\right]\right] z^2 \\ &\leq -2\left[b_0 - \left(\frac{1}{a_0} + \varepsilon\right) c_1 - \left(\frac{d_1 h_0}{c_0} + \varepsilon\right) a_1 - \frac{\alpha^2}{4}\left(a_0 \delta_0\right)\right] z^2 \\ &\leq -2\left[b_0 - \frac{c_1}{a_0} - \frac{d_1 h_0 a_1}{c_0} - \frac{\delta_0}{a_0} - \varepsilon\left(a_1 + c_1\right)\right] z^2 \\ &\leq -2\left[b_0 - \kappa - \varepsilon\left(a_1 + c_1\right)\right] z^2 \leq 0, \end{aligned}$$

$$V_7 \leq -2\left[\alpha a_0 - 1\right] w^2 = -2\varepsilon a_0 w^2 \leq 0,$$

and

$$\begin{aligned} V_8 \;\leq\; & [-2\varepsilon c\left(t\right) + \alpha\rho c_1 + \alpha\rho d_1 \lambda_0] y^2 + [\alpha\rho b_1 + \beta\rho + \mu_1] z^2 \\ & +[\alpha\rho a_1 + \mu_2 + 2\rho] w^2 + [\alpha\rho d_1 \lambda_0 - \mu_1 + \alpha\rho d_1] z_t^2 \\ & +[\alpha\rho a_1 + \alpha\rho b_1 - \mu_2 + \alpha\rho c_1 + \alpha\rho d_1 + 2\rho + \beta\rho] w_t^2 \\ & -2\rho|w w_t| + (\rho - \rho^2) w_t^2 \\ \leq\; & -\left(2\varepsilon c_0 - \alpha\rho c_1 - \alpha\rho d_1 \lambda_0\right) y^2 + \left(\alpha\rho b_1 + \beta\rho + \mu_1\right) z^2 \\ & +\left(\alpha\rho a_1 + 2\rho + \mu_2\right) w^2 + \left(\alpha\rho d_1 \lambda_0 + \alpha\rho d_1 - \mu_1\right) z_t^2 \\ & +\left(\alpha\rho a_1 + \alpha\rho b_1 + \alpha\rho c_1 + \alpha\rho d_1 + \beta\rho + 3\rho - \mu_2\right) w_t^2 \\ & -\rho^2 w_t^2 - 2\rho|w w_t|, \end{aligned}$$

where

$$\lambda_0 = \max\left\{\frac{h_0}{2},\ \left|h_0 - \frac{a_0\delta_0}{d_1}\right|\right\}.$$

By taking

$$\begin{cases} \mu_1 = \alpha\rho d_1\lambda_0 + \alpha\rho d_1, \\ \mu_2 = \alpha\rho a_1 + \alpha\rho b_1 + \alpha\rho c_1 + \alpha\rho d_1 + \beta\rho + 3\rho, \end{cases}$$

we obtain

$$\begin{aligned} V_8 \ \leq\ & -\left(2\varepsilon c_0 - \alpha\rho c_1 - \alpha\rho d_1\lambda_0\right) y^2 + \left(\alpha\rho b_1 + \beta\rho + \mu_1\right) z^2 \\ & + \left(\alpha\rho a_1 + 2\rho + \mu_2\right) w^2 - \rho^2 w_t^2 - 2\rho|ww_t|. \end{aligned}$$

Then we have

$$\begin{aligned} V_5 + V_6 + V_7 + V_8 \ \leq\ & -\rho^2 w_t^2 - 2\rho|ww_t| - \left(2\varepsilon c_0 - \alpha\rho c_1 - \alpha\rho d_1\lambda_0\right) y^2 \\ & -2\left[b_0 - \kappa - \varepsilon\left(a_1 + c_1\right)\right] z^2 \\ & +\left[\rho\left(\alpha b_1 + \beta + \alpha d_1\lambda_0 + \alpha d_1\right)\right] z^2 \\ & -\left(2\varepsilon a_0 - \rho\left(2\alpha a_1 + 5 + \alpha b_1 + \alpha c_1 + \alpha d_1 + \beta\right)\right) w^2, \end{aligned}$$

provided that

$$\begin{aligned} \rho < \min\Bigg\{1,\ \frac{2\varepsilon}{\alpha h_0},\ \frac{2\varepsilon c_0}{\alpha c_1 + \alpha d_1\lambda_0},\ 2\frac{b_0 - \kappa - \varepsilon\left(a_1 + c_1\right)}{\alpha b_1 + \beta + \alpha d_1\lambda_0 + \alpha d_1}, \\ \frac{2\varepsilon a_0}{\alpha(2a_1 + b_1 + c_1 + d_1) + 5 + \beta}\Bigg\}. \end{aligned}$$

Hence, there exists a positive constant D_3 such that,

$$\begin{aligned} V_5 + V_6 + V_7 + V_8 & \leq -2D_3\left(y^2 + z^2 + w^2 + \rho^2 w_t^2 + 2\rho|ww_t|\right) \\ & \leq -2D_3\left(y^2 + z^2 + W^2\right). \end{aligned} \tag{9}$$

Using condition iii) and Lemma 1, we obtain

$$h^2(x) \leq h_0 H(x),$$

consequently,

$$\begin{aligned} |V_9| \ \leq\ & -d'(t)\left[2\beta H(x) + \alpha h_0 y^2 + \left(h^2(x) + y^2\right)\right] \\ & -d'(t)\left[\alpha\left(h^2(x) + z^2\right) + \alpha\rho h^2(x)\right] \\ & +|c'(t)|\left[y^2 + \alpha\left(y^2 + z^2\right)\right] + |b'(t)|\left[\alpha z^2 + \beta y^2\right] \\ & +|a'(t)|\left[z^2 + \beta\left(y^2 + z^2\right)\right] \\ \leq\ & \lambda_2\theta(t)\left(y^2 + z^2 + W^2 + H(x)\right) \\ \leq\ & 2\frac{\lambda_2}{D_0}\theta(t)V, \end{aligned}$$

where we take

$$\begin{aligned}\lambda_2 &= \max\big\{2\beta+(\alpha\rho+\alpha+1)h_0, \alpha h_0+\alpha+2\beta+2, 1+\beta+3\alpha\big\},\\ \theta(t) &= |a'(t)|+|b'(t)|+|c'(t)|-d'(t).\end{aligned}$$

By taking $\dfrac{1}{\eta}=\dfrac{1}{D_0}\lambda_2$, we obtain

$$\begin{aligned}\dot V_{(4)} \leq &- D_3(y^2+z^2+W^2)+\frac{1}{\eta}\theta(t)V\\ &+\big(\beta y+z+\alpha W\big)p(t,x,y,z,w).\end{aligned} \tag{10}$$

From (H2), (8), (10) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\dot U_{(4)} &= \left(\dot V_{(4)}-\frac{1}{\eta}\gamma(t)\,V\right)G(t)\\ &\leq -D_3\left(y^2+z^2+W^2\right)G(t)\\ &\quad \big(\beta y+z+\alpha W\big)p(t,x,y,z,w)\big)G(t)\\ &\leq (\beta|y|+|z|+\alpha|W|)\,|p(t,x,y,z,w)|\\ &\leq D_4\,(|y|+|z|+|W|)\,|e(t)|\\ &\leq D_4\left(3+y^2+z^2+W^2\right)|e(t)|\\ &\leq 3D_4|e(t)|+\frac{D_4}{D_2}U|e(t)|,\end{aligned} \tag{11}$$

where $D_4=\max\{\alpha,\beta,1\}$. Integrating (11) from 0 to t, and using the condition (H2) and Gronwall inequality, we obtain

$$\begin{aligned}U(t,x,y,z,W) &\leq A_0+3D_4\eta_2\\ &\quad +\frac{D_4}{D_2}\int_0^t U\big(s,x(s),y(s),z(s),W(s)\big)|e(s)|ds\\ &\leq \Big(A_0+3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\int_0^t|e(s)|ds}\\ &\leq \Big(A_0+3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\eta_2}=K_1<\infty,\end{aligned} \tag{12}$$

where $A_0=U\big(0,x(0),y(0),z(0),W(0)\big)$. In view of inequalities (8) and (12),

$$(x^2+y^2+z^2+W^2)\leq\frac{1}{D_2}U\leq K, \tag{13}$$

where $K=\frac{K_1}{D_2}$. Clearly, (13) implies that

$$|x(t)|\leq\sqrt{K},\ |y(t)|\leq\sqrt{K},\ |z(t)|\leq\sqrt{K},\ |W(t)|\leq\sqrt{K}\quad\text{for all}\quad t\geq 0.$$

Hence,

$$|x(t)| \leq \sqrt{K},\ |x'(t)| \leq \sqrt{K},\ |x''(t)| \leq \sqrt{K},\ |X'''(t)| \leq \sqrt{K} \quad \text{for all} \quad t \geq 0. \tag{14}$$

Now, we prove the square integrability of solutions and their derivatives. First, from (10) we obtain

$$\dot{V}_{(4)} \leq -D_3(y^2+z^2+w^2) + \frac{1}{\eta}\gamma(t)\,V + \big(\beta y + z + \alpha W\big)p(t,x,y,z,w),$$

thus,

$$\begin{aligned}
\dot{U}_{(4)} &= \left(\dot{V}_{(4)} - \frac{1}{\eta}\gamma(t)\,V\right)G(t) \\
&\leq -D_3\left(y^2+z^2+w^2\right)G(t) \\
&\quad +\big(\beta y + z + \alpha W\big)p(t,x,y,z,w)G(t).
\end{aligned} \tag{15}$$

Now, we define $F_t = F(t,x(t),y(t),z(t),w(t))$ by

$$F_t = U + \sigma\int_0^t \big(y^2(s)+z^2(s)+w^2(s)\big)\,ds,$$

where $\sigma > 0$. It is easy to see that F_t is positive definite, since $U = U(t,x,y,z,w)$ is already positive definite. Using the estimate $e^{-\frac{\eta_1}{\eta}} \leq G(t) \leq 1$ by (H1), and (15), imply

$$\begin{aligned}
\dot{F}_{t(4)} &\leq -D_3\Big(y^2(t)+z^2(t)+w^2(t)\Big)e^{-\frac{\eta_1}{\eta}} \\
&\quad +D_4\Big(|y(t)|+|z(t)|+|W(t)|\Big)|p(t,x,y,z,w)| \\
&\quad +\sigma\Big(y^2(t)+z^2(t)+w^2(t)\Big),
\end{aligned}$$

where D_4 is positive constant. By choosing $\sigma = D_3e^{-\frac{\eta_1}{\eta}}$, we obtain

$$\begin{aligned}
\dot{F}_{t(4)} &\leq D_4\Big(3+y^2(t)+z^2(t)+W^2(t)\Big)|e(t)| \\
&\leq D_4\Big(3+\frac{1}{D_2}U\Big)|e(t)| \\
&\leq 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|.
\end{aligned} \tag{16}$$

Integrating the last inequality (16) from 0 to t, by Gronwall inequality and the condition (H2), we get

$$\begin{aligned}
F_t &\leq F_0 + 3D_4\eta_2 + \frac{D_4}{D_2}\int_0^t F_s|e(s)|ds \\
&\leq \Big(F_0+3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\int_0^t|e(s)|ds} \\
&\leq \Big(F_0+3D_4\eta_2\Big)e^{\frac{D_4}{D_2}\eta_2} = K_2 < \infty.
\end{aligned}$$

Therefore,

$$\int_0^\infty y^2(s)ds < K_2 \quad , \quad \int_0^\infty z^2(s) < K_2 \text{ and } \int_0^\infty w^2(s)ds < K_2,$$

which implies that

$$\int_0^\infty x'^2(s)ds \le K_2 \quad , \quad \int_0^\infty x''^2(s)ds \le K_2 \quad , \quad \int_0^\infty x'''^2(s)ds \le K_2. \tag{17}$$

Next, multiplying (1) by $x(t)$ and integrating by parts from 0 to t, we obtain

$$\int_0^t d(s)x(s)h(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + L_0, \tag{18}$$

where
$I_1(t) = x'(t)X''(t) - x(t)X'''(t) - \int_0^t x''^2(s)ds - \rho \int_0^t x''(s)x''(s-r)ds,$
$I_2(t) = -a(t)x(t)x''(t) + \int_0^t a'(s)x(s)x''(s)ds + \int_0^t a(s)x'(s)x''(s)ds,$
$I_3(t) = -b(t)x(t)x'(t) + \int_0^t b'(s)x(s)x'(s)ds + \int_0^t b(s)x'^2(s)ds,$
$I_4(t) = -\frac{1}{2}c(t)x^2(t) + \frac{1}{2}\int_0^t c'(s)x^2(s)ds,$
$I_5(t) = \int_0^t x(s)p(t, x(s), x'(s), x''(s), x'''(s))ds,$
and

$$L_0 = [X'''(0) + a(0)x''(0) + b(0)x'(0)]x(0) - x'(0)X''(0) + \frac{1}{2}c(0)x^2(0).$$

From (14), (17) and conditions (i) and (H1), we have

$$\begin{aligned}
I_1(t) &\le (2+\rho)K + \frac{1}{2}\rho\int_0^t x''^2(s)ds + \frac{1}{2}\rho\int_0^t x''^2(s-r)ds,\\
&\le (2+\rho)K + \frac{1}{2}\rho\int_0^t x''^2(s)ds\\
&\quad + \frac{1}{2}\rho\int_{-r}^0 x''^2(s)ds + \frac{1}{2}\rho\int_0^{t-r} x''^2(s)ds,\\
I_2(t) &\le a_1K + K\int_0^t |a'(s)|ds + a_1\int_0^t x'(s)x''(s)ds,\\
&\le a_1K + \frac{1}{2}a_1(x'^2(t) - x'^2(0)) + K\int_0^t |a'(s)|ds,\\
I_3(t) &\le b_1K + K\int_0^t |b'(s)|ds + b_1\int_0^t x'^2(s)ds,\\
I_4(t) &\le \frac{1}{2}c_1K + \frac{1}{2}K\int_0^t |c'(s)|ds,\\
I_5(t) &\le \sqrt{K}\int_0^t |e(s)|ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
\lim_{t\to+\infty} I_1(t) &\le (2+\rho)K + \frac{1}{2}\rho K_2 + \frac{1}{2}\rho\int_{-r}^{0} x''^2(s)ds + \frac{1}{2}\rho\int_{0}^{+\infty} x''^2(s)ds,\\
&\le (2+\rho)K + \rho K_2 + \frac{1}{2}\rho K r = L_1,\\
\lim_{t\to+\infty} I_2(t) &\le 2a_1K + K\eta_1 = L_2, \quad \lim_{t\to+\infty} I_3(t) \le b_1K + K\eta_1 + b_1K_2 = L_3,\\
\lim_{t\to+\infty} I_4(t) &\le \frac{1}{2}c_1K + \frac{1}{2}K\eta_1 = L_4, \quad \text{and} \lim_{t\to+\infty} I_5(t) \le \sqrt{K}\eta_2 = L_5.
\end{aligned}$$

Thus,

$$\lim_{t\to+\infty}\big(I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)\big) \le \sum_{i=1}^{5} L_i < \infty. \tag{19}$$

Consequently, (18) and (19), and condition ii) give

$$\int_0^\infty x^2(s)ds \le \frac{1}{d_0\delta}\int_0^\infty d(s)x(s)h(x(s))ds \le \frac{1}{d_0\delta}\sum_{i=0}^{5} L_i < \infty,$$

which completes the proof of the theorem. □

Remark 1 *If $p(t,x,y,z,w)=0$, similarly to above proof, the inequality (9) becomes*

$$V_5+V_6+V_7+V_8 \le -2D_3\left(y^2+z^2+(|w|+\rho|w_t|)^2\right),$$

then,

$$\begin{aligned}
\dot V_{(4)} \le &- D_3(y^2+z^2+(|w|+\rho|w_t|)^2)\\
&+\frac{1}{\eta}\Big(|a'(t)|+|b'(t)|+|c'(t)|-d'(t)\Big)V.
\end{aligned} \tag{20}$$

From (H1), (8), (20) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\dot U_{(4)} &= \left(\dot V_{(4)} - \frac{1}{\eta}\gamma(t)V\right)G(t)\\
&\le -D_3\left(y^2+z^2+(|w|+\rho|w_t|)^2\right)G(t)\\
&\le -\mu\left(y^2+z^2+(|w|+\rho|w_t|)^2\right) \le -\mu\left(y^2+z^2+W^2\right),
\end{aligned}$$

where $\mu = D_3 e^{-\frac{\eta_1}{\eta}}$. It can also be seen that the only solution of system (4) for which $\dot U_{(4)}(t,x,y,z,W)=0$ is the solution $x=y=z=w=0$. The above discussion guarantees that the trivial solution of equation (1) is uniformly asymptotically stable, and the same conclusion as in the proof of Theorem 1 can be drawn for square integrability of solutions of equation (1).

2 Example

We consider the following fourth-order non-autonomous differential equation of neutral type

$$\begin{aligned}
&\left(x(t)+\frac{1}{322}x(t-r)\right)'''' + \left(e^{-t}\sin t+2\right)x''' \\
&+\left(\frac{\sin(t)+7e^{t}+7e^{-t}}{e^{t}+e^{-t}}\right)x''+\left(e^{-2t}\sin^3 t+2\right)x' \\
&+\left(\frac{1}{20\cosh t}+\frac{1+2(1+t^2)}{20(1+t^2)}\right)\left(\frac{x}{x^2+1}+\frac{x}{10}\right) \\
&=\frac{2\sin t}{t^2+(x(t)+x'(t))^2+(x''(t)x'''(t))^2+1}. \qquad (21)
\end{aligned}$$

By taking

$$\begin{aligned}
p(t,x(t),x'(t),x''(t),x'''(t)) &= \frac{2\sin t}{t^2+(x(t)+x'(t))^2+(x''(t)x'''(t))^2+1} \\
&\leq e(t)=\frac{2\sin t}{t^2+1}, \\
h(x) &= \frac{x}{x^2+1}+\frac{x}{10}, \\
h_0-\frac{a_0\delta_0}{d_1}=-\frac{53}{10}\leq h'(x) &= \frac{1-x^2}{(1+x^2)^2}+\frac{1}{10}(x)\leq\frac{h_0}{2}=\frac{11}{10}, \\
a_0=1\leq a(t) &= e^{-t}\sin t+2\leq a_1=3, \\
b_0=\frac{13}{2}\leq b(t) &= \frac{\sin(t)+7e^{t}+7e^{-t}}{e^{t}+e^{-t}}\leq b_1=\frac{15}{2}, \\
c_0=1\leq c(t) &= e^{-2t}\sin^3 t+2\leq c_1=3, \\
d_0=\frac{1}{10}\leq d(t) &= \frac{1}{20\cosh t}+\frac{1+2(1+t^2)}{20(1+t^2)}\leq d_1=\frac{1}{5},
\end{aligned}$$

and

$$\begin{aligned}
b_0=\frac{13}{2} &> \kappa=\frac{d_1h_0a_1}{c_0}+\frac{c_1+\delta_0}{a_0}=\frac{291}{50}, \quad \text{for} \quad \delta_0=\frac{3}{2}, \\
\varepsilon=\frac{1}{20} &< \min\left\{\frac{1}{a_0}, \frac{d_1h_0}{c_0}, \frac{b_0-\kappa}{a_1+c_1}\right\}, \\
\lambda_0=\frac{53}{10} &= \max\left\{\frac{h_0}{2},\left|h_0-\frac{a_0\delta_0}{d_1}\right|\right\},
\end{aligned}$$

we find

$$\begin{aligned}
\alpha = \frac{21}{20} &= \frac{1}{a_0} + \varepsilon, \quad \beta = \frac{49}{100} = \frac{d_1 h_0}{c_0} + \varepsilon, \\
\rho = \frac{1}{322} &< \min\Big\{1, \frac{2\varepsilon}{\alpha h_0}, \frac{2\varepsilon c_0}{\alpha(c_1 + d_1\lambda_0)}, 2\frac{b_0 - \kappa - \varepsilon\,(a_1 + c_1)}{\alpha(b_1 + d_1\lambda_0 + d_1) + \beta}, \\
&\qquad \frac{2\varepsilon a_0}{\alpha(2a_1 + b_1 + c_1 + d_1) + 5 + \beta}\Big\}.
\end{aligned}$$

It follows easily that

$$\begin{aligned}
\int_0^{+\infty} |e\,(t)|\,dt &= \int_0^{+\infty} \left|\frac{2\sin t}{t^2+1}\right| dt \leq \int_0^{+\infty} \frac{2}{t^2+1} dt = \pi, \\
\int_0^{+\infty} |a'\,(t)|\,dt &= \int_0^{+\infty} \left|\,(\cos t)\,e^{-t} - (\sin t)\,e^{-t}\right| dt \leq \int_0^{+\infty} 2e^{-t} dt = 2, \\
\int_0^{+\infty} |b'\,(t)|\,dt &= \int_0^{+\infty} \left|\frac{(e^t + e^{-t})\cos t - (e^t - e^{-t})\sin t}{(e^t + e^{-t})^2}\right| dt \\
&\leq \int_0^{+\infty} \left(\frac{1}{e^t + e^{-t}} + \frac{e^t - e^{-t}}{(e^t + e^{-t})^2}\right) dt \leq \frac{\pi}{2}, \\
\int_0^{+\infty} |c'\,(t)|\,dt &= \int_0^{+\infty} \left|\,3\left(\cos t \sin^2 t\right) e^{-2t} - 2\left(\sin^3 t\right) e^{-2t}\right| dt \\
&\leq \int_0^{+\infty} 5e^{-2t} dt = \frac{5}{2},
\end{aligned}$$

and

$$\int_0^{+\infty} (-d'\,(t))\,dt = \int_0^{+\infty} \frac{1}{20}\left(\frac{\sinh t}{\cosh^2 t} + \frac{2t}{(1+t^2)^2}\right) dt = \frac{1}{10}.$$

Therefore

$$\int_0^{+\infty} (|a'\,(t)| + |b'\,(t)| + |c'\,(t)| - d'\,(t))\,dt < +\infty.$$

Thus, all the assumptions of Theorem 1 hold, so solutions of (21) are bounded and square integrable.

References

[1] Abou-El-Ela, A. M. A.; Sadek, A. I.; Mahmoud, A. M. *On the stability of solutions of certain fourth-order nonlinear nonautonomous delay differential equation.* Int. J. Appl. Math. **22** (2009), no. 2, pp. 245–258.

[2] Andres, J. and Vlček, V.; *On the existence of square integrable solutions and their derivatives to fourth and fifth order differential equations.* Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, **28** (1989), no. 1, pp. 65–86.

[3] Bereketoglu, H., *Asymptotic stability in a fourth order delay differential equation.* Dynam. Systems Appl. **7** (1998), no. 1, pp. 105–115.

[4] Burton T. A., *Stability and periodic solutions of ordinary and functional differential equations* .Mathematics in Science and Engineering, Volume 178, Academic Press, INC, 1985.

[5] Burton T.A., *Volterra Integral and Differential Equations*, Mathematics in Science and Engineering V(202)(2005), 2nd edition.

[6] Cartwright, M. L.; *On the stability of solutions of certain differential equations of the fourth order.* Quart. J. Mech. Appl. Math. **9** (1956), pp. 185–194.

[7] Chin, P. S. M.; *Stability results for the solutions of certain fourth-order autonomous differential equations.*Internat. J. Control. **49** (1989), no. 4, pp. 1163–1173.

[8] El'sgol'ts, L., *Introduction to the Theory of Differential Equations with Deviating Arguments.* Translated from the Russian by Robert J. McLaughlin Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1966.

[9] Ezeilo, J. O. C., A *stability result for solutions of a certain fourth order differential equation.* J. London Math. Soc. **37** (1962), pp. 28–32.

[10] Ezeilo, J. O. C., *On the boundedness and the stability of solutions of some differential equations of the fourth order.* J. Math. Anal. Appl. **5** (1962), pp. 136–146.

[11] Ezeilo, J. O. C., Stability results for the solutions of some third and fourth order differential equations. Ann. Mat. Pura Appl. **66** (1964), no. 4, pp. 233–249.

[12] Ezeilo, J. O. C.; Tejumola, H. O., *On the boundedness and the stability properties of solutions of certain fourth order differential equations.* Ann. Mat. Pura Appl. **95** (1973), no.4, pp. 131–145.

[13] Graef, J. R; Beldjerd, D; Remili, M. *On stability, ultimate boundedness, and existence of periodic solutions of certain third order differential equations with delay.* PanAmerican Mathematical Journal **25** (2015), pp. 82–94.

[14] Greaf, J. R; Oudjedi, L. D, and Remili, M. *Stability and square integrability of solutions of nonlinear third order differential equations.* Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **22** (2015), pp. 313–324.

[15] J. R. Graef, L. D. Oudjedi and M. Remili. *Stability and Square Integrability of solutions to third order neutral delay differential equations* . Tatra Mt. Math. Publ. **71** (2018), pp. 81–97.

[16] Hale, J. K., Theory of Functional Differential Equations. Springer Verlag, New York, 1977.

[17] Hara, T. *On the asymptotic behavior of the solutions of some third and fourth order non- autonomous differential equations.* Publ. RIMS, Kyoto Univ. **9** (1974), pp. 649–673.

[18] Kang, Huiyan; Si, Ligeng, *Stability of solutions to certain fourth order delay differential equations.* Ann. Differential Equations **26** (2010), no. 4, pp. 407–413.

[19] Omeike, P. S. M.; *Boundedness of solutions of the fourth order differential equation with oscillatory restoring and forcing terms.* Analele stiintifice ale univesitatii"AL.I. CUZA" DIN IASI (S.N.) Matematica, Tomul LIV **1** 2008, pp. 187–195.

[20] Oudjedi, L; Beldjerd, D; and Remili, M. *On the stability of solutions for non-autonomous delay differential equations of third-order*, Differential Equations and Control Processes **1** (2014), pp. 22–34.

[21] Rahmane, M; Fatmi, L; and Remili, M. *On stability and boundedness of solutions of fourth-order differential equations with multiple delays.* International Conference on Mathematics and Information Technology (ICMIT) (2017), pp. 376–383.

[22] Rahmane, M; Remili, M. *On stability and boundedness of solutions of certain non autonomous fourth-order delay differential equations.* Acta Universitatis Matthiae Belii, series Mathematics **23** (2015), pp. 101–114.

[23] Remili, M; Beldjerd, D. *A boundedness and stability results for a kind of third order delay differential equations.* Applications and Applied Mathematics **10** (2015), no. 2, pp. 772–782.

[24] Remili, M; Beldjerd, D. *On ultimate boundedness and existence of periodic solutions of kind of third order delay differential equations.* Acta Universitatis Matthiae Belii, series Mathematics (2016), pp. 1–15.

[25] Remili, M; Beldjerd, D. *Stability and ultimate boundedness of solutions of some third order differential equations with delay.* Journal of the Association of Arab Universities for Basic and Applied Sciences **23** (2017), pp. 90–95.

[26] Remili, M; Oudjedi, L. D; and Beldjerd, D. *On the qualitative behaviors of solutions to a kind of nonlinear third order differential equations with delay.* Communications in Applied Analysis 20 (2016), 53-64.

[27] Remili, M; Oudjedi, L. D. *On asymptotic stability of solutions to third order nonlinear delay differential equation.* Filomat **30** (2016), no. 12, pp. 3217–3226.

[28] Remili, M; Rahmane, M. *Sufficient conditions for the boundedness and square integrability of solutions of fourth-order differential equations.* Proyecciones Journal of Mathematics **35** (2016), no. 1, pp. 41–61.

[29] Remili, M; Rahmane, M. *Stability and square integrability of solutions of nonlinear fourth order differential equations.* Bull. Comput. Appl. Math. **4** (2016), no.1, pp. 21–37.

[30] Remili, M; Rahmane, M. *Boundedness and square integrability of solutions of nonlinear fourth order differential equations.* Nonlinear Dynamics and Systems Theory **16** (2016), no. 2, pp. 192–205.

[31] Sadek A I. *On the stability of solutions of certain fourth order delay differential equations.* Applied Mathematics and Computation **148** 2004, no. 2, pp. 587–597.

[32] Sinha, A. S. C., On stability of solutions of some third and fourth order delay-differential equations. Information and Control **23** (1973), pp. 165–172.

[33] Tejumola, H.O.; Tchegnani, B., Stability, boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations. J. Nigerian Math. Soc. **19** (2000), pp. 9–19.

[34] Tunç, C., *On the stability of solutions of non-autonomous differential equations of fourth order with delay.* Funct. Differ. Equ. **17** (2010), no. 1-2, pp. 195–212.

[35] Vlček, V.; *On the boundedness of solutions of a certain fourth-order nonlinear differential equation.* Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **27** (1988), no. 1, pp. 273–288.

[36] Wu, X., Xiong, K.; *Remarks on stability results for the solutions of certain fourth-order autonomous differential equations.* Internat. J. Control. **69** (1998), no. 2, pp 353–360.

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