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On the democratic constant of Haar subsystems in $L_1[0,1]^d$

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Abstract. In this paper, we estimate the democratic constant for the democratic subsystems of the d-dimensional Haar system in $L_1[0,1]^d$.[1]

Key Words: Haar Subsystem, Democratic Constant, Greedy Algorithm in $L_1[0,1]^d$
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Introduction

Let $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ be a normalized basis in a Banach space X. There are many ways to approximate the elements of X by polynomials, using linear combinations of the elements from Ψ. In the late 1990's it became popular to approximate using the Thresholding Greedy Algorithm (TGA, see [1] for details). During their investigations V. Temlyakov and S. Konyagin introduced the term democratic bases (see [2]). Democratic bases have an important role in classification of greedy-type bases. Here, we give a slightly more general definition for democratic systems.

Definition 1 *Let $\Psi = \{\psi_k\}_{k=1}^{+\infty}$ be a normalized system in X. Then Ψ is called democratic in X iff there exists a constant C such that for any two finite subsets A, B of positive integers with equal number of elements ($|A| = |B|$) the following relation holds:*

$$\|\sum_{i\in A}\psi_i\|_X \le C\|\sum_{i\in B}\psi_i\|_X. \tag{1}$$

The smallest C for which this inequality holds is called the democratic constant for Ψ.

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Democratic subsystems of the 1-dimensional and multidimensional Haar systems in $L_1[0,1]$ and $L_1[0,1]^d$ are characterized respectively in [3] and [4]. There are two generalizations of the Haar system in $L_1[0,1]^d$. In [4] the multidimensional Haar system whose all elements have cubic supports is used. Let us recall the definition of that system. The dyadic interval is the interval of type $\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$, with $1 \leq j \leq 2^n$, $n \geq 0$. For a dyadic interval $\mathcal{I} = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right) \subset [0,1)$, we write:

$$r_{\mathcal{I}}^{(0)}(t) = \begin{cases} \frac{1}{|\mathcal{I}|} & : \ t \in \mathcal{I} \\ 0 & : \ t \notin \mathcal{I} \end{cases}, \qquad r_{\mathcal{I}}^{(1)}(t) = \begin{cases} \frac{1}{|\mathcal{I}|} & : \ t \in \left[\frac{j-1}{2^n}, \frac{2j-1}{2^{n+1}}\right) \\ -\frac{1}{|\mathcal{I}|} & : \ t \in \left[\frac{2j-1}{2^{n+1}}, \frac{j}{2^n}\right) \\ 0 & : \ t \notin \mathcal{I} \end{cases}. \tag{2}$$

For dyadic intervals $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_d$ of the same length, the cube

$$\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 \times \ldots \mathcal{I}_d \tag{3}$$

is called a dyadic cube. By $\mathcal{D}^d$ we denote the set of all dyadic cubes of dimension d. To remind the definition of the d-dimensional Haar system, we need one more notation. Denote

$$\mathbb{M} = \mathcal{D}^d \times \{1, 2, \ldots, 2^d - 1\}. \tag{4}$$

To each element $(\mathcal{I}, j) \in \mathbb{M}$, one element of the multidimensional Haar function $h_{\mathcal{I}}^{(j)}$ corresponds, in the following way:

$$h_{\mathcal{I}}^{(j)}(x) = \prod_{i=1}^{d} r_{\mathcal{I}_k}^{(\epsilon_k)}(x_k), \tag{5}$$

where $x = (x_1, x_2, \ldots, x_d) \in [0,1]^d$, and the numbers $\epsilon_k \in \{0,1\}$ are defined by the representation $j = \sum_{k=1}^{d} \epsilon_k 2^{d-k}$. The set of functions $h_{\mathcal{I}}^{(j)}$ together with $h_{[0,1)^d}^{(0)} \equiv 1$ is a d-dimensional Haar system.

Now, for any $\mathcal{I}, \mathcal{J} \in \mathcal{D}^d$ with $\mathcal{J} \subset \mathcal{I}$, denote

$$C(\mathcal{I}, \mathcal{J}) = \{\Delta \in \mathcal{D}^d : \mathcal{J} \subseteq \Delta \subseteq \mathcal{I}\}. \tag{6}$$

$C(\mathcal{I}, \mathcal{J})$ is called a chain (see [5]). The length of the chain $C(\mathcal{I}, \mathcal{J})$ is the number of elements in the chain. Also, we will say that $\mathcal{J}$ is a son of $\mathcal{I}$ iff $\mathcal{J} \subset \mathcal{I}$ and $\mu(\mathcal{J}) = 2^{-d}\mu(\mathcal{I})$.

By the term *complete chain* $C^d(\mathcal{I}, \mathcal{J})$ we mean the following set:

$$C^d(\mathcal{I}, \mathcal{J}) = \{(\Delta, k) \in \mathbb{M} \ : \ \mathcal{J} \subseteq \Delta \subseteq \mathcal{I}, \ 1 \leq k \leq 2^d - 1\}. \tag{7}$$

By the length of chain $C^d(\mathcal{I}, \mathcal{J})$ we mean the length of the chain $C(\mathcal{I}, \mathcal{J})$. Also, for $\mathcal{S} \subset \mathbb{M}$ denote by $D(\mathcal{S})$ the democratic constant for the system $\{h_{\mathcal{I}}^{(j)}\}_{(\mathcal{I},j) \in \mathcal{S}}$ in $L_1[0,1]^d$.

Theorem A *[6] Let $\mathcal{S} \subset \mathbb{M}$ be given and let H be the length of longest complete chain in $\mathcal{S}$ (we assume it is equal to $+\infty$ if there are arbitrarily-long complete chains). Then $\{h_{\mathcal{I}}^{(j)}\}_{(\mathcal{I},j)\in\mathbb{M}}$ is democratic in $L_1[0,1]^d$ if and only if $H < +\infty$.*

From the proof of the theorem, one may conclude that democratic constant satisfies to the condition $D(\mathcal{S}) \leq 2^{Hd}$. In this paper, we improve this result by proving the following theorem.

Theorem 1 *Let $\mathcal{S} \subset \mathbb{M}$ and let $\mathcal{S}$ contain complete chains having maximal length H. Then $D(\mathcal{S}) < 2^d(2^d-1)(H+1)$.*

1 Proof of the result

For $f \in L_1[0,1]^d$ and $\Delta \in \mathcal{D}^d$, we denote

$$\|f\|_\Delta = \int_\Delta |\, f \,|,$$

and for $\mathcal{I} \in \mathcal{D}^d$, we denote

$$P_{\mathcal{I}}(f) = f - \sum_{\mathcal{J}\in\mathcal{D}^d, \mathcal{J}\subseteq\mathcal{I}} \sum_{j=1}^{2^d-1} c_{\mathcal{J}}^{(j)}(f) h_{\mathcal{J}}^{(j)},$$

where $c_{\mathcal{J}}^{(j)}(f), \mathcal{J} \in \mathcal{D}^d, j = 1, \ldots 2^d-1$, are the expansion coefficients of f with respect to the d-dimensional Haar system. Below, we recall several lemmas which will be used to prove our main result.

Lemma 1 ([6], Lemma 1) *Let $f \in L_1[0,1]^d$ and $\mathcal{I}, \mathcal{J} \in \mathcal{D}^d$ be such that $\mathcal{J} \subseteq \mathcal{I}$. Then*

$$\|f\|_{\mathcal{I}} \geq |\, c_{\mathcal{J}}^{(i)}(f) \qquad \text{for all } 1 \leq i \leq 2^d - 1. \tag{8}$$

Lemma 2 ([6], Lemma 2) *Let $f \in L_1[0,1]^d$ and $\mathcal{I}, \mathcal{J} \in \mathcal{D}^d$ be such that:*

1) $|c_{\mathcal{I}}^{(i)}(f)| \leq 1$ for any $(\mathcal{I}, i) \in \mathbb{M}$,

2) $\mathcal{I}$ is a son of $\mathcal{J}$,

3) $c_{\mathcal{J}}^{(i_0)}(f) = 0$ for some $1 \leq i_0 \leq 2^d - 1$.

Then

$$\|P_{\mathcal{I}}(f)\|_{\mathcal{I}} \leq 1 - 2^{-d}. \tag{9}$$

Now, we are ready to present the main lemma of this paper.

Lemma 3 *Let $\Lambda \subset \mathbb{M}$ and let H be the length of the longest complete chain in Λ. Then*

$$\| \sum_{(\mathcal{I},i)\in\Lambda} h_{\mathcal{I}}^{(i)} \| \geq \frac{|\,\Lambda\,|}{2^d(2^d-1)(H+1)}. \tag{10}$$

Proof. Let

$$\{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_k\} = \{\mathcal{I} \ : \ \exists 1 \leq i \leq 2^d - 1, \quad (\mathcal{I}, i) \in \Lambda\}.$$

According to the definition, we have $k \geq \frac{|\Lambda|}{2^d-1}$. Without loss of generality we may assume that

if for $1 \leq i < j \leq k$ one has $\mathcal{I}_j \cap \mathcal{I}_i \neq \emptyset$, then $\mathcal{I}_j \subset \mathcal{I}_i$ and for all $i < s < j$ one has $\mathcal{I}_s \subset \mathcal{I}_i$.

Put $f_0 = 0$ and for $1 \leq s \leq k$ denote

$$f_s = \sum_{i \leq s, j,\ (\mathcal{I}_i, j)\in\Lambda} h_{\mathcal{I}_i}^{(j)}.$$

Again, without loss of generality, we may assume that if $\mathcal{I}_i$ and $\mathcal{I}_j$ are sons of $\mathcal{I}_t$ with respect to $\{\mathcal{I}_i\}$ (with $i < j$) then

$$\|f_t\|_{\mathcal{I}_i} \geq \|f_t\|_{\mathcal{I}_j}.$$

Note, that

$$\| \sum_{(\mathcal{I},i)\in\Lambda} h_{\mathcal{I}}^{(i)} \| = \|f_k\| = \sum_{i=1}^{k} \Big(\|f_i\| - \|f_{i-1}\| \Big).$$

From the monotonicity of the Haar system and definitions of f_i, it follows that all terms in the brackets are non-negative. To complete the proof of the Lemma it remains to show, that at least $\frac{1}{H+1}$ of them have value at least 2^{-d}. Indeed, consider an arbitrary sequence $\mathcal{I}_i, \mathcal{I}_{i+1}, \dots, \mathcal{I}_{i+H}$. By taking into account the definition of H, we can state that they do not form a complete chain in Λ. It follows from the construction that for at least one $j, i < j \leq i + H$ we have one of the following cases:

i) $\mathcal{I}_j \subset \mathcal{I}_{j-1}$ and $\mathcal{I}_j$ is not a son of $\mathcal{I}_{j-1}$. Consider a dyadic cube $\mathcal{J}$ whose son is $\mathcal{I}_j$. According to Lemma 1, we have

$$\|f_j\|_{\mathcal{I}_j} \geq 1$$

and according to Lemma 2, we have (since $(\mathcal{J}, 1) \notin \Lambda$)

$$\|f_{j-1}\|_{\mathcal{I}_j} \leq 1 - 2^{-d}.$$

Since f_{j-1} and f_j coincide outside $\mathcal{I}_j$, we conclude

$$\|f_j\| - \|f_{j-1}\| \geq 2^{-d}.$$

ii) $\mathcal{I}_j$ is a son of $\mathcal{I}_{j-1}$ and for some $t, 1 \le t \le 2^d - 1$ we have $(\mathcal{I}_{j-1}, t) \notin \Lambda$. This case is similar to the previous one. According to Lemma 1, we have

$$\|f_j\|_{\mathcal{I}_j} \geq 1$$

and according to Lemma 2, we have

$$\|f_{j-1}\|_{\mathcal{I}_j} \leq 1 - 2^{-d}.$$

Since f_{j-1} and f_j coincide outside $\mathcal{I}_j$, we conclude

$$\|f_j\| - \|f_{j-1}\| \geq 2^{-d}.$$

iii) $\mathcal{I}_j \cap \mathcal{I}_{j-1} = \emptyset$. In this case we have

$$\|f_{j-1}\|_{\mathcal{I}_j} \leq \frac{1}{2},$$

therefore,

$$\|f_j\|_{\mathcal{I}_j} - \|f_{j-1}\|_{\mathcal{I}_j} \geq \frac{1}{2}.$$

Lemma is proved. □

The proof of the Theorem then easily follows from the lemma.

Proof. Let $A \subset \mathcal{S}$ and $|A| = n$. Note that it is enough to estimate the norm $\| \sum_{(\mathcal{I}_i, j_i) \in A} h_{\mathcal{I}_i}^{(j_i)} \|$.

We have, by Lemma 3, that

$$\| \sum_{(\mathcal{I}_i, j_i) \in A} h_{\mathcal{I}_i}^{(j_i)} \| \geq \frac{n}{2^d(2^d - 1)(H+1)}. \tag{11}$$

Also, by the triangle inequality, we get

$$\| \sum_{(\mathcal{I}_i, j_i) \in A} h_{\mathcal{I}_i}^{(j_i)} \| \leq n. \tag{12}$$

Finally, by (11) and (12), we have that

$$D(S) < 2^d(2^d - 1)(H+1), \tag{13}$$

and this concludes the proof. □

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