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Classifying cubic symmetric graphs of order $18p^2$

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Abstract. A *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. A graph X is called *s-regular* if its automorphism group acts regularly on the set of its s-arcs. In this paper, we classify all connected cubic s-regular graphs of order $18p^2$ for each $s \geq 1$ and each prime p.

Key Words: Symmetric graphs, s-regular graphs regular coverings.
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1 Introduction

Throughout this paper, graphs are assumed to be finite, simple, undirected, and connected. For group theoretic concepts and notation not defined here, we refer the reader to [18, 30]. Given a positive integer n, we shall use the symbol $\mathbb{Z}_n$ to denote the ring of residues modulo n as well as the cyclic group of order n.

For a graph X, we use $V(X)$, $E(X)$, $A(X)$, and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set, and automorphism group, respectively. For u, $v \in V(X)$, uv is the edge incident to u and v in X and $N_X(u)$ is the set of vertices adjacent to u in X. For a subgroup N of $\mathrm{Aut}(X)$, denote by X_N the *quotient graph* of X corresponding to the orbits of N, that is, the graph having the orbits of N as vertices with two orbits adjacent in X_N whenever there is an edge between those orbits in X.

A graph $\widetilde{X}$ is called a *covering* of a graph X with projection $\rho : \widetilde{X} \to X$ if there is a surjection $\rho : V(\widetilde{X}) \to V(X)$ such that $\rho|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in \rho^{-1}(v)$. A covering $\widetilde{X}$ of X with a projection ρ is said to be *regular* (or *k-covering*) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}_K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}_K$ is the composition ρh of ρ and h. If $\widetilde{X}$ is connected, K

becomes the covering transformation group. The *fibre* of an edge or vertex is its preimage under ρ. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre, while every covering transformation maps a fibre onto itself. All fibre-preserving automorphisms from a group are called the *fibre-preserving group.*

Let X be a graph and K be a finite group. We use a^{-1} to denote the reverse arc of an arc a. A *voltage assignment* (or, *K-voltage assignment*) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called *voltages*, and K is called the *voltage group.* The *graph* $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $\rho : X \times_\phi K \to X$, which is called the *natural projection.* By defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, K becomes a subgroup of $\mathrm{Aut}(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a *K-covering.* Conversely, each regular covering $\widetilde{X}$ of X with a covering transformation group K can be derived from a K-voltage assignment. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be *T-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [19] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X.

Let G be a finite group and S be a subset of G such that $1 \notin S$ and $S = S^{-1} = \{s^{-1} | s \in S\}$. The *Cayley graph* $Cay(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{gh |\ g, h \in G, gh^{-1} \in S\}$. A Cayley graph $Cay(G, S)$ is connected if and only if S generates G. It is well known that $\mathrm{Aut}(Cay(G, S))$ contains the *right regular representation* $R(G)$ of G, the acting group of G by right multiplication, which is regular on vertices. A Cayley graph $Cay(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(Cay(G, S))$. A graph X is isomorphic to a Cayley graph on G if and only if $\mathrm{Aut}(X)$ has a subgroup isomorphic to G, acting regularly on vertices (see [5, Lemma 16.3]).

An *s-arc* in a graph X is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. A graph X is said to be *s-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric.* A symmetric graph X is said to be *s-regular* if $\mathrm{Aut}(X)$ acts regularly on the set of s-arcs in X. Tutte [28, 29] showed that every cubic symmetric graph is s-regular for some $1 \le s \le 5$.

Following the pionerring article of Tutte [28], cubic symmetric graphs have been extensively studied over decades by many authors. Most of these works have been focused on classification results and constructions of infinite families. For example, Djoković and Miller [10] constructed an infinite family of cubic 2-regular graphs, and Conder and Preager [8] constructed two infinite families of cubic s-regular graphs for $s = 2$ or 4. Marušič and Pisanski [22] classifed cubic s-regular Cayley graphs on the dihedral groups. Feng and Kwak [15] classified cubic symmetric graphs of order a small number times a prime or a prime square. Following this, classifications of cubic s-regular graphs of orders $4p^i, 6p^i, 8p^i, 10p^i, 16p^i, 12p^i$, $22p^i, 36p^i$ for $i = 1, 2$ and prime integer p were presented in [12, 16, 3, 25, 1, 27, 4]. Furthermore, cubic s-regular graphs of orders $2p^3, 14p, 6p^3, 28p$, and $4m$ where m is an odd integer were classified in [17, 24, 2, 20, 7]. In this paper, we obtain a classification of cubic symmetric graphs of order $18p^2$.

2 Preliminaries

Let G be a group. The *center* $Z(G)$ is the set of elements which commute with every element of G, and it is a normal subgroup of G. If $a, b \in G$, the *commutator* of a and b, denoted by $[a, b]$, is $[a, b] = aba^{-1}b^{-1}$. The *derived subgroup* of G, denoted by $G^{'}$, is the subgroup of G generated by all the commutators.

The following proposition is a straightforward consequence of Theorems 10.1.5 and 10.1.6 of [26].

Proposition 1 *Let G be a finite group and p be a prime. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.*

By [21, Theorem 9], we have the following proposition.

Proposition 2 *Let X be a connected symmetric graph of prime valency and G be an s-arc-transitive subgroup of $Aut(X)$ for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular, and G/N is an s-arc-transitive subgroup of $Aut(X_N)$. Furthermore, X is a regular covering of X_N with the covering transformation group N.*

Let $X = Cay(G, S)$ be a Cayley graph on a group G with respect to a subset S of G. Set $A =$Aut(X) and Aut(G, S)= $\{\alpha \in \text{Aut}(G) | S^\alpha = S\}$.

Proposition 3 *[31, Proposition 1.5] The Cayley graph X is normal if and only if $A_1 =$Aut(G, S) where A_1 is the stabilizer of the vertex $1 \in V(X) = G$ in A.*

The Pappus graph F_{18} is illustrated in Figure 1. It is known that F_{18} is the unique connected cubic symmetric graph of order 18 (see [6]). Let T be a spanning tree of F_{18}, as shown by dart lines in Figure 1.

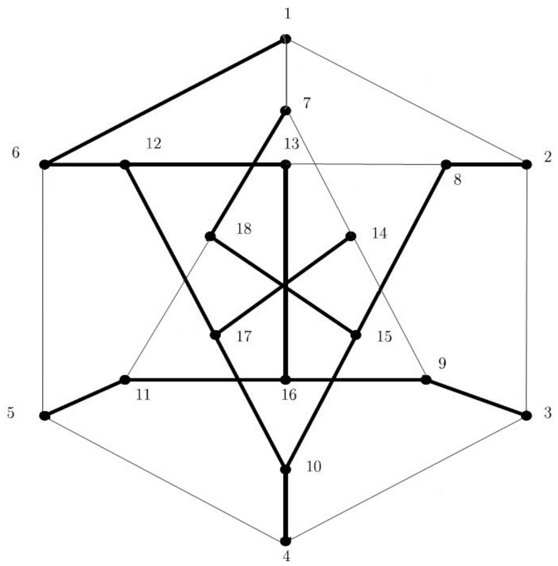

Figure 1: The Pappus graph

Let $p \geq 7$ be a prime and $EF_{18p^2} = X \times_\phi \mathbb{Z}_p^2$ where the voltage assignment $\phi : A(X) \to \mathbb{Z}_p^2$ is defined by $\phi = 0$ on T and $\phi = (1,0), (0,1), (0,-1), (0,1), (0,-1), (1,1), (0,0), (1,0), (0,0)$, and $(1,0)$ on the cotree arcs $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, $(5,6)$, $(1,7)$, $(7,14)$, $(13,8)$, $(14,9)$, and $(11,18)$, respectively. By [23, Theorem 3.1], we have the following lemma.

Lemma 1 *Let $p \geq 7$ be a prime and let X be a connected $\mathbb{Z}_p^2$-covering of the graph F_{18} whose fibre-preserving group is arc-transitive. Then X is isomorphic to the 2-regular graph EF_{18p^2}.*

Let p be a prime. It is easy to check that the equation

$$x^2 + x + 1 = 0 \tag{1}$$

has no solution in the ring $\mathbb{Z}_{3p^2}$ for $p = 3$. The following result determines the solutions of equation 1 in $\mathbb{Z}_{3p^2}$ when $p \neq 3$.

Lemma 2 *Let $p \neq 3$ be a prime. Then there exists an element $k \in \mathbb{Z}_{3p^2}$ solving equation 1 if and only if k is an element of order 3 in $\mathbb{Z}_{3p^2}^*$.*

Proof. Suppose first that $k \in \mathbb{Z}_{3p^2}$ such that $k^2 + k + 1 = 0$. Then $k \neq 1$, and since $k^3 - 1 = (k-1)(k^2+k+1) = 0$, it follows that k is an element of order 3 in $\mathbb{Z}_{3p^2}^*$.

Conversely, suppose that k is an element of order 3 in $\mathbb{Z}_{3p^2}^*$. Then $k \neq 1$ and $k^3 = 1$. It follows that $(k-1)(k^2+k+1) = 0$. If $k-1$ is divisible by 3, then k^2+k+1 is also divisible by 3. Thus, in order to prove $k^2+k+1 = 0$, it suffices to show $(k-1,\ p) = 1$. Assume that $k-1$ is divisible by p, that is,

$k \equiv 1 (\mathrm{mod}\ p)$. Then $k^2 + k + 1 \equiv 3 \pmod p$, and since $p \neq 3$, we conclude that $k^2 + k + 1$ is coprime with p, which implies that $k \equiv 1 \pmod{p^2}$. Let $k = tp^2 + 1$. Then $k^3 = t^3 p^6 + 1$, and since $k^3 = 1$, we have $t \equiv 0 (\mathrm{mod}\ 3)$. Hence $k = 1$, a contradiction. This completes the proof of lemma. □

Let p be a prime such that $p \equiv 1 \pmod 3$. Since $\mathbb{Z}^*_{3p^2} \cong \mathbb{Z}_2 \times \mathbb{Z}_{p(p-1)}$, by Lemma 2, there are exactly two elements of order 3, say k and k^2 in $\mathbb{Z}^*_{3p^2}$, solving equation 1. Set $V(K_{3,3}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ to be the vertex set of the complete bipartite graph $K_{3,3}$ with partite sets $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$. The graphs CF_{18p^2} and $\overline{CF}_{18p^2}$ are defined to have the same vertex set $V(CF_{18p^2}) = V(\overline{CF}_{18p^2}) = V(K_{3,3}) \times \mathbb{Z}_{3p^2}$ and edge sets

$$\begin{aligned} E(CF_{18p^2}) =& \{\{(\mathbf{a}, i), (\mathbf{x}, i)\}, \{(\mathbf{a}, i), (\mathbf{y}, i)\}, \{(\mathbf{a}, i), (\mathbf{z}, i)\}, \\ & \{(\mathbf{b}, i), (\mathbf{x}, i + k + 1)\}, \{(\mathbf{b}, i), (\mathbf{y}, i)\}, \{(\mathbf{b}, i), (\mathbf{z}, i + 1)\}, \\ & \{(\mathbf{c}, i), (\mathbf{x}, i - 1)\}, \{(\mathbf{c}, i), (\mathbf{y}, i - k - 1)\}, \{(\mathbf{c}, i), (\mathbf{z}, i)\} \big| i \in \mathbb{Z}_{3p^2}\}, \end{aligned}$$

$$\begin{aligned} E(\overline{CF}_{18p^2}) =& \{\{(\mathbf{a}, i), (\mathbf{x}, i)\}, \{(\mathbf{a}, i), (\mathbf{y}, i)\}, \{(\mathbf{a}, i), (\mathbf{z}, i)\}, \\ & \{(\mathbf{b}, i), (\mathbf{x}, i + k^2 + 1)\}, \{(\mathbf{b}, i), (\mathbf{y}, i)\}, \{(\mathbf{b}, i), (\mathbf{z}, i + 1)\}, \\ & \{(\mathbf{c}, i), (\mathbf{x}, i - 1)\}, \{(\mathbf{c}, i), (\mathbf{y}, i - k^2 - 1)\}, \{(\mathbf{c}, i), (\mathbf{z}, i)\} \big| i \in \mathbb{Z}_{3p^2}\}, \end{aligned}$$

respectively. The graph $\overline{CF}_{18p^2}$ is obtained by replacing k with k^2 in each edge of CF_{18p^2}. It is easy to see that CF_{18p^2} and $\overline{CF}_{18p^2}$ are cubic and bipartite.

Lemma 3 *The graphs CF_{18p^2} and $\overline{CF}_{18p^2}$ are isomorphic.*

Proof. Let p be a prime such that $p \equiv 1 \pmod 3$. To show $CF_{18p^2} \cong \overline{CF}_{18p^2}$, we define a map α from CF_{18p^2} to $\overline{CF}_{18p^2}$ by

$$\begin{array}{lll} (\mathbf{a}, i) \longmapsto (\mathbf{a}, ki), & (\mathbf{b}, i) \longmapsto (\mathbf{c}, ki), & (\mathbf{c}, i) \longmapsto (\mathbf{b}, ki), \\ (\mathbf{x}, i) \longmapsto (\mathbf{x}, ki), & (\mathbf{y}, i) \longmapsto (\mathbf{z}, ki), & (\mathbf{z}, i) \longmapsto (\mathbf{y}, ki), \end{array}$$

where $i \in \mathbb{Z}_{3p^2}$. Clearly,

$$\begin{aligned} N_{CF_{18p^2}}((\mathbf{b}, i)) &= \{(\mathbf{y}, i),\ (\mathbf{x}, i + k + 1),\ (\mathbf{z}, i + 1)\}, \\ N_{\overline{CF}_{18p^2}}((\mathbf{b}, i)^\alpha) &= N_{\overline{CF}_{18p^2}}((\mathbf{c}, ki)) \\ &= \{(\mathbf{x}, ki - 1),\ (\mathbf{y}, ki - k^2 - 1),\ (\mathbf{z}, ki)\}. \end{aligned}$$

By Lemma 2, $k^2 + k + 1 = 0$. Using this property, one can easily show that

$$[N_{CF_{18p^2}}((\mathbf{b}, i))]^\alpha = N_{\overline{CF}_{18p^2}}((\mathbf{b}, i)^\alpha).$$

Similarly,

$$[N_{CF_{18p^2}}((\mathbf{u}, i))]^\alpha = N_{\overline{CF}_{18p^2}}((\mathbf{u}, i)^\alpha),$$

for $\mathbf{u} = \mathbf{a},\ \mathbf{c}$. It follows that α is an isomorphism from CF_{18p^2} to $\overline{CF}_{18p^2}$, because the graphs are bipartite. □

In view of [13, Theorem 1.1] and Lemmas 2 and 3, we have the following lemma.

Lemma 4 *Let p be a prime and let X be a connected $\mathbb{Z}_{3p^2}$-covering of the complete bipartite graph $K_{3,3}$ whose fibre-preserving group is arc-transitive. Then $p \equiv 1$ (mod 3), and X is isomorphic to the 1-regular graph CF_{18p^2}.*

3 Main result

In this section, we shall determine all connected cubic symmetric graphs of order $18p^2$ for each prime p. We start with the following useful lemma.

Lemma 5 *Let $p \geq 7$ be a prime and let X be a connected cubic symmetric graph of order $18p^2$. Then Aut(X) has a normal Sylow p-subgroup.*

Proof. Let X be a cubic graph satisfying the assumptions and let $A = \mathrm{Aut}(X)$. Since X is symmetric, by Tutte [28], X is s-regular for some $1 \leq s \leq 5$. Thus, $|A| = 2^s \cdot 3^3 \cdot p^2$. Let N be a minimal normal subgroup of A.

Suppose that N is unsolvable. Then $N = T \times T \times ... \times T = T^k$ where T is a non-abelian simple group. Since $p \geq 7$ and A is a $\{2,\ 3,\ p\}$-group, by [18, pp. 12-14] and [9], T is one of the following groups: $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$, and $\mathrm{PSL}(3,3)$ with orders $2^3 \cdot 3 \cdot 7$, $2^3 \cdot 3^2 \cdot 7$, $2^4 \cdot 3^2 \cdot 17$, and $2^4 \cdot 3^3 \cdot 13$, respectively. Since 2^6 does not divide $|A|$, one has $k = 1$, and hence $p^2 \nmid |N|$. It follows that N has more than two orbits on $V(X)$. By Proposition 2, N is semiregular on $V(X)$, which implies that $|N| \big| 18p^2$, a contradiction. Thus, N is solvable.

For any prime divisor q of $|A|$, let $O_q(A)$ be the maximal normal q-subgroup of A. By Proposition 2, $O_q(A)$ is semiregular on $V(X)$, and the quotient graph $X_{O_q(A)}$ is a cubic symmetric graph with $A/O_q(A)$ as an arc-transitive subgroup of $\mathrm{Aut}(X_{O_q(A)})$. The semiregularity of $O_q(A)$ implies that $|O_q(A)| \big| 18p^2$. If $O_2(A) \neq 1$, then $O_2(A) \cong \mathbb{Z}_2$, and hence $X_{O_2(A)}$ has odd order and valency 3, a contradiction. Thus, $O_2(A) = 1$, and by the solvability of N, either $O_3(A) \neq 1$ or $O_p(A) \neq 1$. Let $O_3(A) \neq 1$. Then $|O_3(A)| = 3$ or 9, and so $X_{O_3(A)}$ is a cubic symmetric graph of order $6p^2$ or $2p^2$. Let $M/O_3(A)$ be a minimal normal subgroup of $A/O_3(A)$. Then, by the same argument as above, one may show that $M/O_3(A)$ is solvable and hence elementary abelian. Since $O_3(A/O_3(A)) = 1$, $M/O_3(A)$ is a 2- or p-group. For the former by Proposition 2, the quotient graph X_M would have be a cubic graph of odd order, a contradiction. Thus, $M/O_3(A)$ is a p-group. Since $p \geq 7$, by Sylow Theorem, M has a normal Sylow p-subgroup which is characteristic in M and hence normal in A because $M \lhd A$. Therefore, $O_p(A) \neq 1$.

Let $Q := O_p(A)$. To prove the lemma, we only need to show that $|Q| = p^2$. Suppose to the contrary that $|Q| = p$. Let $C = C_A(Q)$. Then $Q \leq Z(C)$. By Proposition 1, $p \nmid |C' \bigcap Z(C)|$, which implies that $C' \bigcap Q = 1$. This forces $p^2 \nmid |C'|$, and hence C' has more than two orbits on $V(X)$. Note that C' is characteristic in C, and $C \trianglelefteq A$. Then $C' \lhd A$. By Proposition 2, C' is semiregular on $V(X)$, and the quotient graph $X_{C'}$ is a cubic symmetric graph. Therefore, we can conclude that $|C'|\big|9p$. Let P be a Sylow p-subgroups of A. As P is abelian, $P < C$, and so PC'/C' is a sylow p-subgroup of C/C'. Note that C/C' is abelian. Then PC'/C' is characteristic in C/C', and since $C/C' \trianglelefteq A/C'$, we have $PC'/C' \trianglelefteq A/C'$. Hence $PC' \lhd A$. Clearly, $|PC'| = tp^2$ where $t \mid 9$. Since $p \geq 7$, P is normal in PC'. This implies that P is characteristic in PC'. Thus, $P \lhd A$, which is contrary to $|Q| = p$. □

We are now ready to prove the main result of this paper.

Theorem 1 *Let X be a connected cubic symmetric graph of order $18p^2$ where p is a prime. Then X is 1-, 2-, or 3-regular. Furthermore,*

(i) X is 1-regular if and only if X is isomorphic to one of the graphs F_{162B} and CF_{18p^2} where $p \equiv 1$ (mod 3);

(ii) X is 2-regular if and only if X is isomorphic to one of the graphs F_{74}, F_{162A}, F_{450}, and EF_{18p^2} where $p \geq 7$;

(iii) X is 3-regular if and only if X is isomorphic to the graph F_{162C}.

Proof. Let X be a connected cubic symmetric graph of order $18p^2$ and let A =Aut(X). Then $|A| = 2^s \cdot 3^3 \cdot p^2$ for some integer $1 \leq s \leq 5$. For $p = 2$ or 5, by [6], there is only one connected cubic symmetric graph of order $18p^2$, which is the 2-regular graph F_{18p^2}, and for $p = 3$, there are three connected cubic symmetric graphs of order 18×3^2, which are the 1-regular graph F_{162B}, the 2-regular graph F_{162A}, and the 3-regular graph F_{162C}. Thus, we may assume that $p \geq 7$. Let P be a Sylow p-subgroup of A. Then by Lemma 5, P is normal in A. Since $|P| = p^2$, we have $P \cong \mathbb{Z}_p^2$ or $\mathbb{Z}_{p^2}$.

Suppose first that $P \cong \mathbb{Z}_p^2$. Then by Proposition 2, X is a $\mathbb{Z}_p^2$-covering of the Pappus graph, and since $P \lhd A$, the symmetry of X means that the fibre-preserving group is arc-transitive. By Lemma 3, $X \cong EF_{18p^2}$ because $p \geq 7$.

Now Suppose that $p \cong \mathbb{Z}_{p^2}$. Then by Proposition 2, the quotient graph X_P is a cubic symmetric graph, and A/P is an arc-transitive subgroup of Aut(X_P). Let $C = C_A(P)$. Clearly, $P \leq C$. Suppose that $P = C$. Then by [26, Theorem 1.6.13], A/P is isomorphic to a subgroup of Aut$(P) \cong \mathbb{Z}_{p(p-1)}$, which implies that A/P is abelian. Since A/P is transitive on $V(X_P)$, it follows by [30, Proposition 4.4] that A/P is regular on $V(X_P)$. Consequently, $|A| = 18p^2$, which is impossible. Thus, $P < C$. Let M/P be a minimal normal subgroup of A/P contained in C/P. Since $|A/P| = 2^s \cdot 3^3$, by [26, Theorem 8.5.3], M/P is solvable and hence elementary abelian 2- or 3-group. If M/P is a 2-group, then by Proposition 2, X_M is a cubic symmetric graph of odd order, a contradiction. Thus, M/P is a 3-group. Let Q be a Sylow 3-subgroup of M. Then $M = PQ$, implying $M = P \times Q$ because $Q < C$. Hence, Q is characteristic in M, and since $M \lhd A$, we have $Q \lhd A$. Again, by Proposition 2, Q is semiregular on $V(X)$. Note that Q is isomorphic to M/P, and M/P is elementary abelian. Then by the semiregularity of Q, we have $Q \cong \mathbb{Z}_3$ or $\mathbb{Z}_3^2$.

Let $Q \cong \mathbb{Z}_3^2$. Then, as above, the quotient graph X_Q is a cubic symmetric graph order $2p^2$, and A/Q is an arc-transitive subgroup of $\mathrm{Aut}(X_Q)$. Note that $|A/Q| = 2^s \cdot 3 \cdot p^2$. Then the Sylow p-subgroup PQ/Q of A/Q is also a Sylow p-subgroup of $\mathrm{Aut}(X_Q)$, implying the Sylow p-subgroups of $\mathrm{Aut}(X_Q)$ are cyclic. Since $p \geq 7$, by [11, Lemma 3.4] and [14, Theorem 3.5], X_Q is a normal cubic 1-regular Cayley graph on dihedral group D_{2p^2}. Thus, $A/Q = \mathrm{Aut}(X_Q)$, and A has a normal subgroup G, such that G/Q acts regularly on $V(X_Q)$. Consequently, G is regular on $V(X)$, and hence X is a normal cubic 1-regular Cayley graph on G. Let $X = Cay(G, S)$. Since X has valency 3, S contains at least one involution. By Proposition 3, $\mathrm{Aut}(G, S)$ is transitive on S, which implies that S consists of three involutions, and by the connectivity of X, G can be generated by three involutions. Clearly, $M < G$. Since $G/Q \cong D_{2p^2}$, we conclude that $G \cong \mathbb{Z}_3^2 \times D_{2p^2}$ or $\mathbb{Z}_3 \times D_{6p^2}$, which is impossible because in each case G can not be generated by involutions.

Thus $Q \cong \mathbb{Z}_3$, and so $M \cong \mathbb{Z}_{3p^2}$. By Proposition 2, X is a $\mathbb{Z}_{3p^2}$-covering of the bipartite graph $K_{3,3}$, and the normality of M implies that the fibre-preserving group is the automorphism group A of X, so it is arc-transitive. By Lemma 4, X is isomorphic to CF_{18p^2} where $p \equiv 1 \pmod 3$. □

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