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Hermitian Toeplitz determinants for the class $\mathcal{S}$ of univalent functions

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Abstract. Introducing a new method, we give sharp estimates of the Hermitian Toeplitz determinants of third order for the class $\mathcal{S}$ of functions univalent in the unit disc. The new approach is also illustrated on some subclasses of the class $\mathcal{S}$.

Key Words: univalent, Hermitian Toeplitz determinant of second order, Hermitian Toeplitz determinant of third order, class $\mathcal{U}$, convex functions
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1 Introduction

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and normalized such that $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S} \subset \mathcal{A}$ be the class of univalent functions in the unit disc $\mathbb{D}$ (functions that are analytic, one-on-one and onto).

For functions $f \in \mathcal{A}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and positive integers q and n, the Toeplitz matrix is defined by

$$T_{q,n}(f) = \begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \dots & a_n \end{bmatrix},$$

where $\overline{a}_k = \overline{a_k}$. Thus, the second Toeplitz determinant is

$$|T_{2,1}(f)| = 1 - |a_2|^2$$

and the third is

$$|H_{3,1}(f)| = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 2\,\mathrm{Re}\,(a_2^2\overline{a}_3) - 2|a_2|^2 - |a_3|^2 + 1. \qquad (1)$$

The concept of Toeplitz matrices plays an important role in functional analysis, applied mathematics as well as in physics and technical sciences (for more details, see [28]).

If a_n is real, then the Toeplitz matrix $T_{q,n}(f)$ is an Hermitian one, i.e., it is equal to its conjugate transpose: $T_{q,n}(f) = \overline{[T_{q,n}(f)]^T}$. Determinants of Hermitian matrices are real numbers. Additionally, if $n = 1$, the determinant $|T_{q,1}(f)|$ is rotationally invariant, i.e., for any real θ, the determinants $|T_{q,1}(f)|$ and $|T_{q,1}(f_\theta)|$ of the Hermitian Toeplitz matrices of functions $f \in \mathcal{A}$ and $f_\theta(z) := e^{-i\theta} f(e^{i\theta} z)$ have the same values.

Recently, various problems regarding upper bounds, preferably sharp, of determinants involving coefficients of univalent functions, were rediscovered and attract significant interest. The highest focus is on the Hankel determinant and valuable references with overview of older results and the new ones are [2, 3, 6–9, 11, 18–25, 27–29].

Naturally rises the question of finding lower and upper bound estimates of the determinant of the Hermitian Toeplitz matrices for the class of univalent functions and its subclasses. This problem was initiated by Cudna et al. ([4]) and Kowalczyk et al. ([5]), and sharply solved in [4] for the classes of starlike and convex functions of order α, $0 \le \alpha < 1$, defined respectfully by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\, z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha,\, z \in \mathbb{D} \right\}.$$

For finding sharp estimates of the Hermitian Toeplitz determinant of second order, it is enough to know sharp estimate for the second coefficient. The same question for the third order determinant turns out to be more complicated.

In this paper, we introduce new method for obtaining estimates of the Hermitian Toeplitz determinants of third order and receive sharp result for the general class $\mathcal{S}$ of univalent functions.

We illustrate the new method also on the class of convex functions, receiving the same sharp result as in [4]. In a similar manner, we study classes

$$\mathcal{U}_s(\lambda) = \left\{ f \in \mathcal{U}(\lambda) : \frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)} \right\} \quad (0 < \lambda \le 1)$$

and

$$\mathcal{G}(\delta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] < 1 + \frac{\delta}{2},\, z \in \mathbb{D} \right\} \quad (0 < \delta \le 1),$$

where

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda,\ z \in \mathbb{D} \right\} \quad (0 < \lambda \le 1)$$

and "$\prec$" denotes the ususal subordination. Class $\mathcal{U}(\lambda)$ is not included in the class of starlike functions $\mathcal{S}^* := \mathcal{S}^*(0)$, nor vice versa (see [13, 14]). Therefore, estimates for $\mathcal{S}^*$ can not be transferred to the class $\mathcal{U}(\lambda)$. Sharp upper bound of the Hankel determinant of second and third order for the class $\mathcal{U} := \mathcal{U}(1)$ are given in [19].

One can note that $\mathcal{U} = \mathcal{U}_s(1)$, since for all functions f from $\mathcal{U}$, $z/f(z) \prec 1/(1+z)^2$ (see [12]), while the general implication

$$f \in \mathcal{U}(\lambda) \quad \Rightarrow \quad \frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)},$$

$0 < \lambda < 1$, was claimed in [15], but proven to be wrong in [10] by giving a counterexample.

2 Main results

We start with the following sharp estimates for the Hermitian Toeplitz determinants.

Theorem 1 *If $f \in \mathcal{S}$, then*

$$-3 \le |T_{2,1}(f)| \le 1 \quad \textit{and} \quad -1 \le |T_{3,1}(f)| \le 8.$$

All inequalities are sharp.

Proof. From the Bieberbach's theorem ([1]) we have $|a_2| \le 2$ for all functions from $\mathcal{S}$ with Koebe's function $k(z) = z/(1-z)^2 = \sum_{k=1}^{\infty} kz^k$ as an extremal one. Now both estimates for $|T_{2,1}(f)|$, together with their sharpness, directly follow.

We continue with study of the third Toeplitz determinant. Since for the class $\mathcal{S}$, $|a_3 - a_2^2| \le 1$ (see [25, p.5]), then

$$|a_2|^4 + |a_3|^2 - 2\,\mathrm{Re}\,(a_2^2\overline{a_3}) = |a_3 - a_2^2|^2 \le 1, \tag{2}$$

and hence,

$$2\,\mathrm{Re}\,(a_2^2\overline{a_3}) \ge |a_2|^4 + |a_3|^2 - 1.$$

Now, by using (1) we have

$$|T_{3,1}(f)| \ge (|a_2|^2 - 1)^2 - 1 \ge -1,$$

which is sharp as the function $f_1(z) = z/(1 - z + z^2) = z + z^2 - z^4 - \cdots$ shows.

As for the upper bound of $|T_{3,1}(f)|$, from (1), by using that $\mathrm{Re}\,(a_2^2 a_3) \le |a_2|^2|a_3|$, we obtain

$$|T_{3,1}(f)| \le -|a_3|^2 + 2|a_2|^2|a_3| - 2|a_2|^2 + 1 =: \varphi(|a_3|),$$

where

$$\varphi(t) = -t^2 + 2|a_2|^2 t - 2|a_2|^2 + 1 \quad \text{and} \quad 0 \le t = |a_3| \le 3.$$

We need to find $\max \varphi(t)$ for $t \in [0, 3]$.

In that sense we have two cases. The first one is $0 \le |a_2|^2 \le 3$, i.e., $0 \le |a_2| \le \sqrt{3}$, when

$$\max \varphi(t) = \varphi(|a_2|^2) = (|a_2|^2 - 1)^2 \le 4.$$

The second case is $3 \le |a_2|^2 \le 4$, i.e., $\sqrt{3} \le |a_2| \le \sqrt{2}$, when

$$\max \varphi(t) = \varphi(3) = 4|a_2|^2 - 8 \le 8.$$

Therefore, $\max \varphi(t) = 8$, when $t \in [0, 3]$.

The result is sharp as the Koebe function $k(z)$ shows. □

Remark 1

(*i*) *The same result as in Theorem 1 holds for the class* $\mathcal{S}^* = \mathcal{S}^*(0)$ *(see Corollary 1 and Corollary 3 in [4]).*

(*ii*) *The same result as in Theorem 1 holds for the class* $\mathcal{U} = \mathcal{U}(1)$ *since* $\mathcal{U} \subset \mathcal{S}$ *and both extremal functions* f_1 *and* k *belong to* $\mathcal{U}$*.*

Remark 2 *It is evident that for applying the method used in the proof of Theorem 1 on other classes of univalent functions, it is enough to know the sharp estimates for* $|a_2|$, $|a_3|$ *and* $|a_3 - a_2^2|$ *and apply them on*

$$|T_{2,1}(f)| = 1 - |a_2|^2 \tag{3}$$

and on

$$|T_{3,1}(f)| \le -|a_3|^2 + 2|a_2|^2|a_3| - 2|a_2|^2 + 1 =: \varphi(|a_3|), \tag{4}$$

where $\varphi(t) = -t^2 + 2|a_2|^2 t - 2|a_2|^2 + 1$ *and* $t = |a_3|$.

In the sense of Remark 2, for the class $\mathcal{U}_s(\lambda)$, using the sharp estimates

$$|a_2| \le 1 + \lambda, \quad |a_3| \le 1 + \lambda + \lambda^2 \quad \text{and} \quad |a_3 - a_2^2| \le \lambda, \tag{5}$$

(estimate $|a_2| \le 1 + \lambda$ is sharp on the whole class $\mathcal{U}(\lambda)$) given in [16], [17] and [10], we receive the following theorem.

Theorem 2 *If $f \in \mathcal{U}(\lambda)$, then*

$$-\lambda(2+\lambda) \le |T_{2,1}(f)| \le 1,$$

and if additionally $f \in \mathcal{U}_s(\lambda)$, then

$$-\lambda^2 \le |T_{3,1}(f)| \le \begin{cases} 1, & 0 \le \lambda \le \lambda_0, \\ \lambda^2(1+\lambda)(3+\lambda), & \lambda_0 \le \lambda \le 1, \end{cases}$$

where $\lambda_0 = 0.44762\ldots$ is the positive real root of the equation

$$\lambda^2(1+\lambda)(3+\lambda) - 1 = 0.$$

All inequalities are sharp.

Proof. The estimates for the second Hermitian Toeplitz determinant follow directly from (3) and (5), and they are sharp due to the functions $f_3(z) = z$ and

$$\begin{aligned} f_4(z) &= \frac{z}{1-(1+\lambda)z+\lambda z^2} = \frac{z}{(1-z)(1-\lambda z)} \\ &= z + (1+\lambda)z^2 + (1+\lambda+\lambda^2)z^3 + \cdots . \end{aligned}$$

For the lower bound of the third Hermitian Toeplitz determinant, from (4) and (5), we have

$$|T_{3,1}(f)| \ge \left(|a_2|^2 - 1\right)^2 - \lambda^2 \ge -\lambda^2,$$

with sharpness for the function $f_2(z) = z/(1-z+\lambda z^2) = z + z^2 + (1-\lambda)z^3 + \cdots$. Function f_2 is analytic on $\mathbb{D}$ since $1 - z + \lambda z^2$ equals zero on the unit disk only when $\lambda = 0$ and $\lambda = -2$.

For the upper bound of $|T_{3,1}(f)|$ we consider two cases.

In the first one, when $0 \le |a_2|^2 \le 1+\lambda+\lambda^2$, the vertex of the parabola $\varphi(t)$ is obtained for $t = |a_2|^2$ and lies in the range of $t = |a_3|$. Thus,

$$\begin{aligned} |T_{3,1}(f)| &\le \max \varphi(t) = \varphi(|a_2|^2) = \left(|a_2|^2 - 1\right)^2 \\ &\le \begin{cases} 1, & |a_2|^2 \le 2, \\ \lambda^2(1+\lambda)^2, & 2 \le |a_2|^2 \le 1+\lambda+\lambda^2, \end{cases} \\ &= \begin{cases} 1, & 0 < \lambda \le (\sqrt{5}-1)/2, \\ \lambda^2(1+\lambda)^2, & (\sqrt{5}-1)/2 \le \lambda \le 1. \end{cases} \end{aligned}$$

Similarly, in the second case, $1+\lambda+\lambda^2 \le |a_2|^2 \le (1+\lambda)^2$, we have that the vertex lies on the right of the range of $t = |a_3|$. Thus,

$$|T_{3,1}(f)| \le \max \varphi(t) = \varphi(1+\lambda+\lambda^2) \le \lambda^2(1+\lambda)(3+\lambda).$$

By using all these facts, we conclude that

$$|T_{3,1}(f)| \leq \begin{cases} 1, & 0 < \lambda \leq \lambda_0, \\ \lambda^2(1+\lambda)(3+\lambda), & \lambda_0 \leq \lambda \leq 1, \end{cases}$$

where $\lambda_0 = 0.44762\ldots$ is the positive real root of the equation

$$\lambda^2(1+\lambda)(3+\lambda) - 1 = 0.$$

The upper bound of the third order determinant is also sharp with extremal function f_3 when $0 < \lambda \leq \lambda_0$ and f_4 when $\lambda_0 \leq \lambda \leq 1$. $\square$

For $\lambda = 1$, we receive the following corollary with the same estimates as for the class $\mathcal{S}$ already discussed in Remark 1(ii).

Corollary 1 *If $f \in \mathcal{U} \equiv \mathcal{U}(1)$, then $-3 \leq |T_{2,1}(f)| \leq 1$, and if $f \in \mathcal{U}_s \equiv \mathcal{U}_s(1)$, then $-1 \leq |T_{3,1}(f)| \leq 8$. All inequalities are sharp.*

We conclude with two more applictions of Remark 2.

Theorem 3 *If $f \in \mathcal{C} := \mathcal{C}(0)$, then $0 \leq |T_{3,1}(f)| \leq 1$. The estimate is sharp.*

Proof. For the class $\mathcal{C}$ of convex functions, we know that

$$|a_3 - a_2^2| \leq \frac{1}{3}(1 - |a_2|^2)$$

(see [26]). Therefore, from (4) we have

$$|T_{3,1}(f)| \geq \frac{8}{9}\left(1 - |a_2|^2\right)^2 \geq 0.$$

The function $f_5(z) = z/(1-z) = z + z^2 + z^3 + \cdots$ shows that this result is sharp.

On the other hand, since $0 \leq |a_2| \leq 1 = \max |a_3|$, we have

$$|T_{3,1}(f)| \leq \max \varphi(t) = \varphi(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1,$$

with equality for $f_3(z) = z$.

Therefore, $0 \leq |T_{3,1}(f)| \leq 1$, which is the same result as in Corollary 6 in [4]. $\square$

Theorem 4 *If $f \in \mathcal{G} := \mathcal{G}(1)$, then we have sharp estimates*

$$\frac{1}{2} \leq |T_{3,1}(f)| \leq 1.$$

Proof. For the class $\mathcal{G}$, we have

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{6} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{1}{4}$$

(see [16]). Then

$$|T_{3,1}(f)| \geq \left(1 - |a_2|^2\right)^2 - \frac{1}{16} \geq \left(\frac{3}{4}\right)^2 - \frac{1}{16} = \frac{1}{2}.$$

The result is sharp as the function $f_6(z) = z - z^2/2$ shows.

As for the upper bound, for $0 \leq |a_2|^2 \leq 1/6 = \max |a_3|$, we have

$$\max \varphi(t) = \varphi(|a_2|^2) = (|a_2|^2 - 1)^2 \leq 1,$$

while for $1/6 \leq |a_2|^2 \leq 1/4$,

$$\max \varphi(t) = \varphi\left(\frac{1}{6}\right) = \frac{35}{36} - \frac{5}{3}|a_2|^2 \leq \frac{35}{36} - \frac{5}{3} \cdot \frac{1}{6} = \frac{25}{36},$$

which implies that $|T_{3,1}(f)| \leq \max \varphi(t) = 1$ for $0 \leq t = |a_3| \leq 1/6$. The result is sharp for $f_3(z) = z$.

This result can be easily generalized on the class $\mathcal{G}(\delta)$ using the sharp estimates require for the method given in [16]. $\square$

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