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A Combinatorial Interpretation of the Padovan Generalized Polynomial Sequence

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Abstract. We investigate a combinatorial interpretation of the Padovan polynomial sequence, also addressing its polynomial extensions. We thus include the Tridovan polynomial sequence, Tetradovan polynomial sequences, leading up to the Z-dovan polynomial generalization.

Key Words: Combinatorics, Generalization, Padovan Polynomial Sequence
Mathematics Subject Classification 2020: 11B37, 11B39

Introduction

In order to investigate new interpretations for the Padovan polynomial sequence, a study of the generalized Padovan sequence combinatorial models is carried out. Recent works on recurrent numerical sequences involving new combinatorial approaches were reviewed in the literature, introducing new approaches to visualizing the sequence terms in a combinatorial manner [1, 2, 4].

Spreafico [5] defined the notion of a board for studying of combinatorial interpretations of sequences. Benjamin and Quinn [1] conducted a study of the combinatorial approach to the Fibonacci and Lucas sequences, presenting evidence through tiling and the study of identities. Tedford [6] performed a combinatorial interpretation for the Padovan sequence using dominoes and triminoes. Vieira, Alves and Catarino [9] accomplished the combinatorial interpretation by inserting a black square and new configuration rules for the generalized Padovan sequence.

In this work, we introduce a combinatorial interpretation for the generalized Padovan polynomial sequence, based on the works of Tedford [6] and Vieira [8]. Vieira [8] also carried out a generalization of these numbers, defining the polynomial sequence from Tridovan, Tetradovan to its generalization called Z-dovan.

1 The generalized Padovan polynomial sequences

Vieira [8] defined (s,t)-Padovan sequence, generalizing the recurrence coefficients. Based on the same idea as for the Fibonacci polynomial sequence [3], one has the definition of the Padovan polynomial sequence assigning values to s and t, with $s = x$ and $t = 1$. Delaying the sequence is also necessary for obtaining other initial values.

Definition 1 *The Padovan polynomial sequence satisfies the following recurrence formula: for $n \in \mathbb{N}$,*

$$P_n(x) = xP_{n-2}(x) + P_{n-3}(x),$$

$n \geqslant 3$, where $P_n(x)$ is the n-th term of the Padovan polynomial sequence and $P_0(x) = 1$, $P_1(x) = 0$, $P_2(x) = x$ are the initial terms.

The first terms of $P_n(x)$ for a given x are presented in Table 1.

Vieira [8] and Vieira and Alves [7] generalized the order of the Padovan sequence numbers, thus we have the polynomial Padovan sequence extension. Particularly, the Tridovan polynomial sequence, is a fourth order extension of the Padovan polynomial sequence.

Definition 2 *The Tridovan polynomial sequence satisfies the following recurrence formula: for $n \in \mathbb{N}$,*

$$T_n(x) = x^2T_{n-2}(x) + xT_{n-3}(x) + T_{n-4}(x),$$

$n \geqslant 4$, where $T_n(x)$ is the n-th term of the Tridovan polynomial sequence with initial terms $T_0(x) = 1$, $T_1(x) = 0$, $T_2(x) = x^2$ and $T_3(x) = x$.

Table 1 shows the first ten terms of the Tridovan polynomial sequence.

The Tetradovan sequence is a fifth-order polynomial Padovan sequence extension [7].

Definition 3 *The Tetradovan sequence satisfies the following recurrence relation:*

$$Te_n(x) = x^3Te_{n-2}(x) + x^2Te_{n-3}(x) + xTe_{n-4}(x) + Te_{n-5}(x),$$

$n \geqslant 5$, where $Te_n(x)$ is the n-th term of the Tetradovan polynomial sequence with the following initial values $Te_0(x) = 1$, $Te_1(x) = 0$, $Te_2(x) = x^3$, $Te_3(x) = x^2$ and $Te_4(x) = x^6 + x$.

n	$P_n(x)$	$T_{(n)}(x)$	$Te_{(n)}(x)$
0	1	1	1
1	0	0	0
2	x	x^2	x^3
3	1	x	x^2
4	x^2	x^4+1	x^6+x
5	$2x$	$2x^3$	$2x^5+1$
6	x^3+1	x^6+3x^2	x^9+3x^4
7	$3x^2$	$3x^5+2x$	$3x^8+4x^3$
8	x^4+3x	x^8+6x^4+1	$x^{12}+6x^7+3x^2$
9	$4x^3+1$	$4x^7+7x^3$	$4x^{11}+10x^6+2x$

Table 1: First terms of the Padovan, Tridovan and Tetratovan polynomial sequences. Source: Prepared by the authors.

See Table 1 for the first ten terms of the Tetradovan polynomial sequence.

From the extensions of the previous sequences, one can obtain the generalization of the Padovan polynomial sequence for an extension of order z, called the Z-dovan polynomial with $Z = z - 1$.

Definition 4 *The Z-dovan polynomial sequence satisfies the following recurrence formula:*

$$Z_{n+z}(x) = \sum_{i=0}^{z-2} Z_{n+i}x^{z-i-2},$$

$n \geqslant z$, $n \in \mathbb{N}$, *where* Z_n *is the* n*-th term of the Z-dovan polynomial sequence.*

2 A combinatorial approach to the generalized Padovan polynomial sequences

We now proceed to introduce a combinatorial interpretation for the Padovan numbers. This interpretation is defined as the set of tilings that are considered based on an n-size board. A tiling is a row of n squares where different objects in a domino shape must fit.

Theorem 1 *The number* $p_n(x)$ *of ways(tilings) to fill a Padovan polynomial board of size* n *with: gray dominoes of size 1 x 2 weighing* x *and white triminoes 1 x 3 weighing* 1 *a is determined as* $p_n(x) = P_n(x)$ *for* $n \geqslant 1$.

Proof. It is necessary to show that $p_n(x)$ satisfies the same recursive definition and initial conditions as $P_n(x)$.

It is not difficult to see that

$$\begin{aligned}
p_1(x) &= 0 = P_1(x),\\
p_2(x) &= x = P_2(x),\\
p_3(x) &= 1 = P_3(x),\\
p_4(x) &= x^2 = P_4(x).
\end{aligned}$$

For that, $\{p_n(x)\}_{n\geq 1}$ has the same initial conditions as $\{P_n(x)\}_{n\geq 1}$. Suppose $n \geq 3$ and divide $p_n(x)$ into two subsets as follows. Let D be the subset of n tiles that end in dominoes and T be the subset of n tiles that end in triminoes. Then $p_n(x) = D \cup T$ and $|p_n(x)| = |D| + |T|$. Since $|D| = p_{n-2}(x)$ and $|T| = p_{n-3}(x)$, this implies that $p_n(x) = p_{n-2}(x) + p_{n-3}(x)$. Thus, $p_n(x)$ satisfies the same recursive formula as $P_n(x)$. Therefore, $p_n(x) = P_n(x)$ for all $n \geq 1$. $\square$

Figure 1: Padovan polynomial tiling for $p_1(x), p_2(x), p_3(x)$ and $p_4(x)$.
Source: Prepared by the authors.

A demonstration of the statement of the Theorem 1 is presented on Fig. 2. For cases from $p_1(x)$ to $p_9(x)$. For the initial case $p_1(x)$, the value 0 is obtained, which is the term $P_1(x)$ of the Padovan polynomial sequence. For $p_2(x)$, we have x tilings, representing the member $P_2(x)$ of the sequence. For $p_3(x)$, there is 1 tiling, which is $P_3(x)$, etc.

Next we present the Tridovan's polynomial combinatorial interpretation.

Theorem 2 *The number $t_n(x)$ of ways to cover an n-board with 1 x 2 gray dominoes weighing x^2, 1 x 3 white triminoes weighing x, and 1 x 4 green tetraminos weighing* 1 *is given by $t_n(x) = T_n(x)$, $n \geq 1$.*

The proof is analogous to the proof of Theorem 1. See Fig.3 for the cases from $t_1(x)$ to $t_9(x)$.

Now, let us consider the Tetratovan's polynomial combinatorial interpretation.

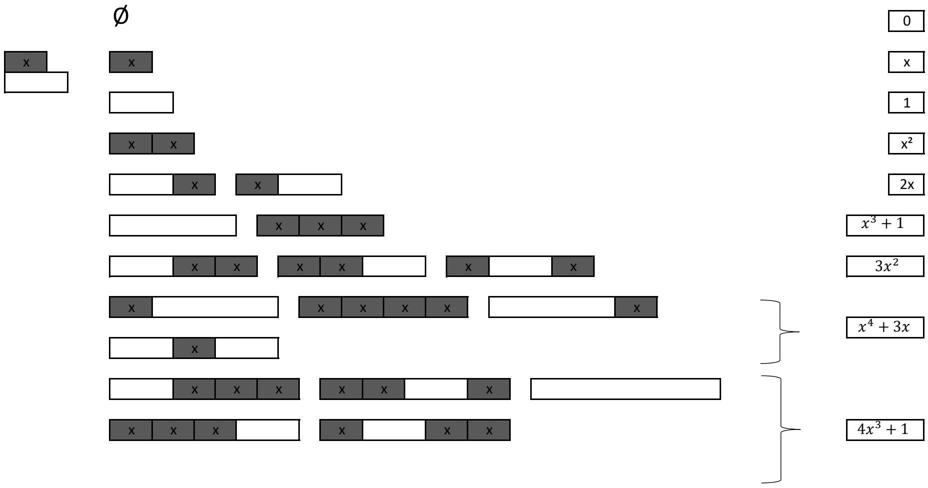

Figure 2: Padovan polynomial tiling from $p_1(x)$ to $p_9(x)$.
Source: Prepared by the authors.

Theorem 3 *The number $te_n(x)$ of ways to cover an n-board with 1 x 2 gray dominoes weighing x^3, 1 x 3 white triminoes weighing x^2, 1 x 4 green tetraminos weighing x and 1 x 5 yellow pentominoes weighing* 1*, is given by $te_n(x) = Te_n(x)$, $n \geq 1$.*

The proof is analogous to the proof of Theorem 1 and is exemplified on Fig. 4.

Finally, we introduce the polynomial combinatorial interpretation of Z-dovan.

Let $z_n(x)$ be the number of ways to fill an n-Z-dovan polynomial board with the following tiles: gray dominoes 1 x 2 weight referring to the coefficient of the first term of the recurrence, white triminos 1 x 3 of weight referring to the coefficient of the second term of the recurrence, etc, and lilac rectangles 1 x z weighing 1.

In Fig. 5, the term $z_n(x)$ is demonstrated which is the number of tile shapes on the n-board corresponding to the aforementioned rules.

Corollary 1 *For $n \geq 2$,*

$$z_n(x) = Z_n(x),$$

where z_n is the number of shapes to fill the 1 x n board of Z-dovan and $Z_n(x)$ is the nth term of the polynomial sequence of Z-dovan.

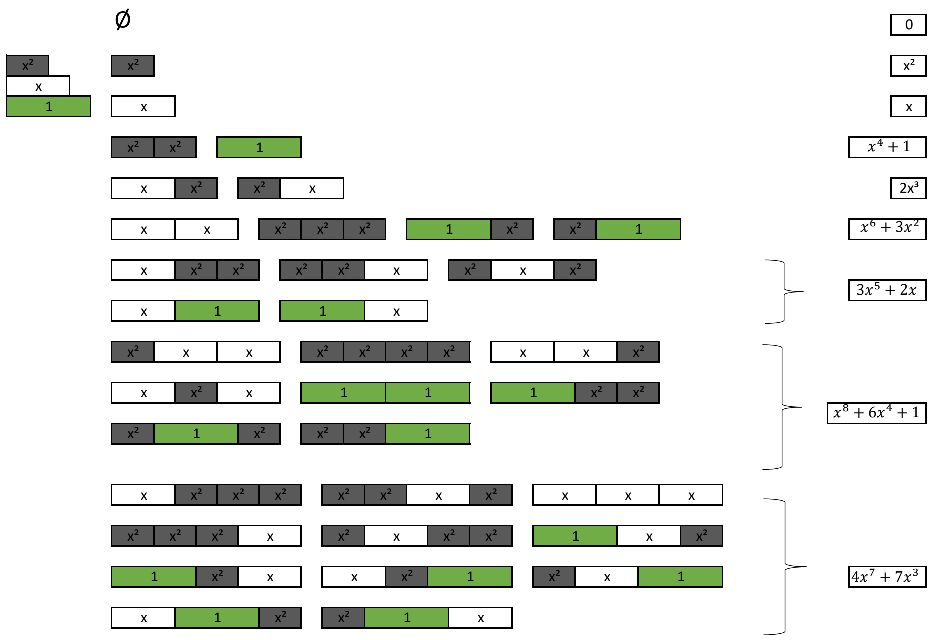

Figure 3: Tridovan polynomial tiling from $t_1(x)$ to $t_9(x)$.
Source: Prepared by the authors.

Note that the insertion of the tiles are given according to the order of the sequence. Thus, for the Padodvan polynomial sequence (order 3) two pieces (dominoes and triminoes) were established; for the Tridovan polynomial sequence (order 4), 3 pieces were available (dominoes, triminos and tetraminos); for the Tetradovan polynomial sequence (order 5), 4 pieces were available (dominoes, triminos, tetraminos and pentominoes). With this, it is clear that for a sequence regarding the weights, they are inserted according to the sequence recurrence formula, allowing the smaller piece to present the weight with greater power. The larger piece has the weight 0 of order z, there will be $z-1$ pieces available.

3 Conclusions

Starting from the combinatorial interpretation of the Padovan polynomial sequence, it was possible to extend it to the Tridovan and Tetradovan polynomial sequences, with a generalization to the Z-dovan sequence (order z).

This approach allowed a visualization of terms by tilings on the n-size board and with respective defined weights and pieces.

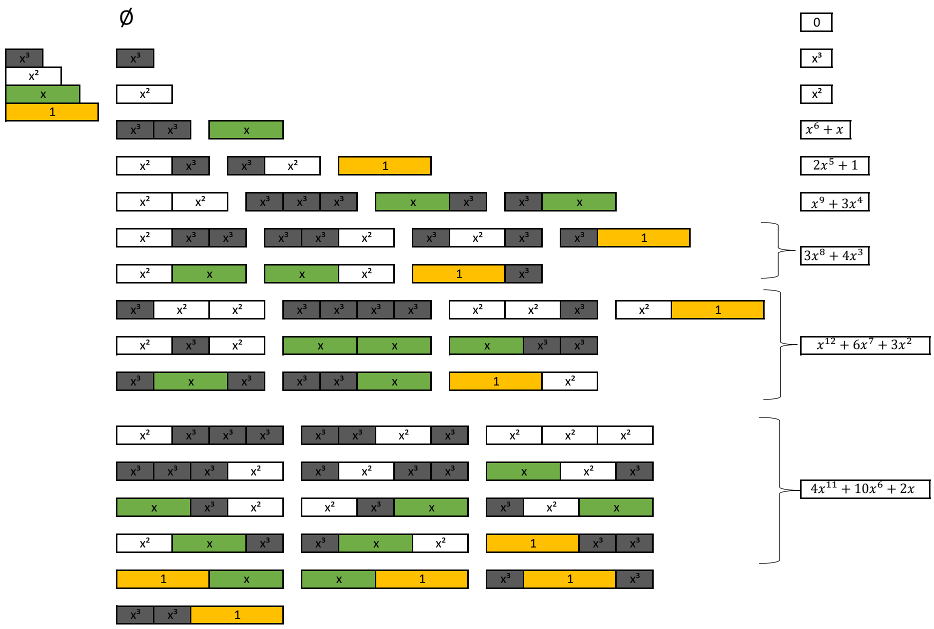

Figure 4: Tetradovan tiling from $te_1(x)$ to $te_9(x)$.
Source: Prepared by the authors.

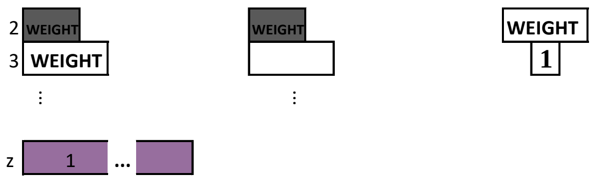

Figure 5: Configuration of Z-dovan polynomial tilings.
Source: Prepared by the authors.

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