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The Generating Function of a Bi-Periodic Leonardo Sequence

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Abstract. In “A note on bi-periodic Leonardo sequence”, the generating function for a certain bi-periodic Leonardo sequence is claimed. In this note, based on the literature, we establish the correct identity. Possible periodic extensions for the Leonardo sequence are discussed, opening new avenues for results in the area.

Key Words: Leonardo Numbers, Generating Functions, Hessenberg Matrices, Recurrence Relations, Determinant
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1 The Leonardo sequence and a bi-periodic extension

The Leonardo sequence (Le_n) is defined by the inhomogeneous recurrence relation

$$\mathrm{Le}_n = \mathrm{Le}_{n-1} + \mathrm{Le}_{n-2} + 1 \quad \text{for } n \geqslant 2,$$

with initial conditions

$$\mathrm{Le}_0 = \mathrm{Le}_1 = 1\,.$$

Alternatively, it can be defined by the homogeneous recurrence relation

$$\mathrm{Le}_n = 2\,\mathrm{Le}_{n-1} - \mathrm{Le}_{n-3} \quad \text{for } n \geqslant 3. \tag{1}$$

The first few terms of the Leonardo sequence are

$$1,\ 1,\ 3,\ 5,\ 9,\ 15,\ 25,\ 41,\ 67,\ 109,\ 177,\ 287,\ 465,\ \ldots.$$

In the recent work [6], Catarino and Spreafico proposed a bi-periodic extension of Leonardo sequence, say $(\mathrm{Le}_n^{(a,b)})$, defined by the recurrence relations

$$\mathrm{Le}_n^{(a,b)} = \begin{cases} a\,\mathrm{Le}_{n-1}^{(a,b)} + \mathrm{Le}_{n-2}^{(a,b)} + a\,, & \text{if } n \text{ is even,} \\ b\,\mathrm{Le}_{n-1}^{(a,b)} + \mathrm{Le}_{n-2}^{(a,b)} + b\,, & \text{if } n \text{ is odd,} \end{cases} \tag{2}$$

for $n \geqslant 2$, with initial conditions

$$\mathrm{Le}_0^{(a,b)} = 2a - 1 \quad \text{and} \quad \mathrm{Le}_1^{(a,b)} = 2ab - 1\,.$$

The first terms of this bi-periodic version of the Leonardo sequence are

$2a-1$, $2ab-1$, $2a(ab+1)-1$, $2a(ab+1)-1$, $2a(ab(ab+3)+1)-1$, $2ab(ab+1)(ab+3)-1$, $\ldots$

In Lemma 2 of [6], the authors deduce the single inhomogeneous recurrence relation

$$\mathrm{Le}_n^{(a,b)} = (ab+2)\,\mathrm{Le}_{n-2}^{(a,b)} - \mathrm{Le}_{n-4}^{(a,b)} + ab \tag{3}$$

for $n \geq 4$. This means that if we subtract $\mathrm{Le}_n^{(a,b)}$ to $\mathrm{Le}_{n-1}^{(a,b)}$ as in (3), we obtain a homogeneous recurrence relation of 5th order.

Lemma 1 *The bi-periodic Leonardo sequence* ($Le_n^{(a,b)}$) *satisfies the homogeneous recurrence relation*

$$Le_n^{(a,b)} = Le_{n-1}^{(a,b)} + (ab+2)\,Le_{n-2}^{(a,b)} - (ab+2)Le_{n-3}^{(a,b)} - Le_{n-4}^{(a,b)} + Le_{n-5}^{(a,b)}\,. \tag{4}$$

In [6, Theorem 3], the authors claim that the generating function for the bi-periodic Leonardo sequence ($\mathrm{Le}_n^{(a,b)}$) is

$$\frac{(1-z)(4a-2-ab-abz-(2a-1)z^2+(2ab-1)(ab+2)z^3)+ab}{(1-z)(1-(ab+2)z^2+z^4)}\,.$$

In this note we show that this statement is not accurate. We provide the correct generating function and corresponding proof based on existing literature. This will be done in the next section where the main tools will also be provided. Our approach is of matricial nature. In the last section, we propose a new bi-periodic Leonardo sequence and possible extensions.

2 Recurrence relations, Hessenberg matrices, and generating functions

Let us consider the sequence (a_n) defined by the homogeneous recurrence relation

$$a_n = p_{n,n-1}\,a_{n-1} + \cdots + p_{n,n-r}\,a_{n-r} \tag{5}$$

for $n > r$, with given initial conditions

$$a_1 = b_1\,, \ldots\,, a_r = b_r. \tag{6}$$

It is well-known that (a_n) can be written explicitly as the determinant of a Hessenberg matrix, namely,

$$a_n = \det \begin{pmatrix} b_1 & b_2 & \cdots & b_r & & & \\ -1 & 0 & \cdots & 0 & p_{r+1,1} & & \\ & -1 & \ddots & \vdots & \vdots & \ddots & \\ & & \ddots & 0 & \vdots & & \ddots & \\ & & & -1 & p_{r+1,r} & & & p_{n,n-r} \\ & & & & -1 & \ddots & & \vdots \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & -1 & p_{n,n-1} \end{pmatrix}. \tag{7}$$

For details and general applications, the reader is referred to [14, 18, 22]. A general result can be found, for example, in [21, Theorem 4.20]. Furthermore, if we replace the -1's of the subdiagonal of the Hessenberg matrix defined in (7) by 1's, then a_n is the permanent of such matrix (cf. [8]).

Using classical results on Hessenberg matrices it is possible to find in many instances explicit formulas or the generating function for (7). For example, in [11] we have the following result.

Theorem 1 *The generating function of the sequence (d_n) defined by the determinants*

$$d_n = \det \begin{pmatrix} b_0 & b_1 & \cdots & b_k & & & \\ -1 & c_1 & c_2 & \cdots & c_\ell & & \\ & -1 & c_1 & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c_\ell \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & -1 & c_1 & c_2 \\ & & & & & -1 & c_1 \end{pmatrix}_{(n+1)\times(n+1)}$$

is

$$\sum_{n=0}^{\infty} d_n z^n = \frac{b_0 + b_1 z + \cdots + b_k z^k}{1 - c_1 z - \cdots - c_\ell z^\ell}.$$

The reader is also invited to read [13] for an earlier look and historical context and [15] for other extensions. Note that we assume $k < \ell$.

From (7) and taking into account the recurrence relation (4), we obtain for $\mathrm{Le}_{n-1}^{(a,b)}$

$$\begin{vmatrix} \mathrm{Le}_0^{(a,b)} & \mathrm{Le}_1^{(a,b)} & \mathrm{Le}_2^{(a,b)} & \mathrm{Le}_3^{(a,b)} & \mathrm{Le}_4^{(a,b)} & & & \\ -1 & 0 & 0 & 0 & 0 & 1 & & \\ & -1 & 0 & 0 & 0 & -1 & \ddots & \\ & & -1 & 0 & 0 & -(ab+2) & \ddots & 1 \\ & & & -1 & 0 & ab+2 & \ddots & -1 \\ & & & & -1 & 1 & \ddots & -(ab+2) \\ & & & & & \ddots & \ddots & ab+2 \\ & & & & & & -1 & 1 \end{vmatrix}_{n\times n}$$

for $n \geqslant 1$. Using elementary operations over the first 5 columns of the Hessenberg matrix defined above, the determinant equals

$$\begin{vmatrix} 2a-1 & 2a(b-1) & 2-a(b+2) & 2a & -1 & & & \\ -1 & 1 & ab+2 & -(ab+2) & -1 & 1 & & \\ & -1 & 1 & ab+2 & -(ab+2) & -1 & \ddots & \\ & & -1 & 1 & ab+2 & -(ab+2) & \ddots & 1 \\ & & & -1 & 1 & ab+2 & \ddots & -1 \\ & & & & -1 & 1 & \ddots & -(ab+2) \\ & & & & & \ddots & \ddots & ab+2 \\ & & & & & & -1 & 1 \end{vmatrix}.$$

From Theorem 1, we finally get

$$\sum_{n=0}^{\infty} \mathrm{Le}_n^{(a,b)} z^n = \frac{2a-1+2a(b-1)z+(2-a(b+2))z^2+2az^3-z^4}{1-z-(ab+2)z^2+(ab+2)z^3+z^4-z^5}.$$

A straightforward factorization of the denominator lead us to the following theorem.

Theorem 2 *The generating function for the bi-periodic Leonardo sequence* $(Le_n^{(a,b)})$ *is*

$$\sum_{n=0}^{\infty} Le_n^{(a,b)} z^n = \frac{2a-1+2a(b-1)z+(2-a(b+2))z^2+2az^3-z^4}{(1-z)(1-(ab+2)z^2+z^4)}.$$

3 An extension for the bi-periodic Leonardo sequence

The main motivation for introducing the bi-periodic Leonardo sequence $(\mathrm{Le}_n^{(a,b)})$ is the bi-periodic Horadam sequence (h_n) defined by the homogeneous recurrence relations

$$h_n = \begin{cases} ah_{n-1} + h_{n-2}\,, & \text{if } n \text{ is even,} \\ bh_{n-1} + h_{n-2}\,, & \text{if } n \text{ is odd} \end{cases}$$

for $n \geqslant 2$, with initial conditions $h_0 = a$ and $h_1 = ab + 1$. This sequence can be deduced by the determinantal identity

$$h_{n-1} = \det \begin{pmatrix} a & 1 & & \\ -1 & b & 1 & \\ & -1 & a & \ddots \\ & & \ddots & \ddots \end{pmatrix}_{n\times n}$$

for $n \geqslant 1$, and can be determined in terms of Chebyshev polynomials of the second kind. These matrices are called bi-periodic, and the study of their determinants, called *continuants*, dates back to the 1940's (or, perhaps, even earlier) in the context of physics and chemistry with the article [20]. See also [7].

A natural and full extension to this sequence is (f_n) defined by the recurrence relations

$$f_n = \begin{cases} a_1 h_{n-1} + a_2 h_{n-2}\,, & \text{if } n \text{ is even,} \\ b_1 h_{n-1} + b_2 h_{n-2}\,, & \text{if } n \text{ is odd.} \end{cases}$$

The tridiagonal matrix associated with this sequence is called 2-Toeplitz and its study, extensions, and applications have been meticulously carried out over time (cf., e.g., [1, 2, 3, 4, 5, 9, 10, 12, 16, 17, 19]).

Consequently, we proposed a first new bi-periodic Leonardo sequence, say generically $(\tilde{\mathrm{Le}}_n^{(a,b)})$, defined by the recurrence relations

$$\tilde{\mathrm{Le}}_n^{(a,b)} = \begin{cases} a_1\, \tilde{\mathrm{Le}}_{n-1}^{(a,b)} + a_2\, \tilde{\mathrm{Le}}_{n-2}^{(a,b)} + a_1 a_2\,, & \text{if } n \text{ is even,} \\ b_1\, \tilde{\mathrm{Le}}_{n-1}^{(a,b)} + b_2\, \tilde{\mathrm{Le}}_{n-2}^{(a,b)} + b_1 b_2\,, & \text{if } n \text{ is odd} \end{cases}$$

for $n \geqslant 2$, with initial conditions

$$\tilde{\mathrm{Le}}_0^{(a,b)} = a_1 - a_2 + a_1 a_2 \quad \text{and} \quad \tilde{\mathrm{Le}}_1^{(a,b)} = b_1(a_1 - a_2 + a_1 a_2) + b_2(b_1 - 1).$$

Another possibility which might be of interest is when the recurrence relations satisfy

$$\tilde{\mathrm{Le}}_n^{(a,b)} = \begin{cases} a_1\, \tilde{\mathrm{Le}}_{n-1}^{(a,b)} + a_2\, \tilde{\mathrm{Le}}_{n-2}^{(a,b)} + a_1 + (1 - a_2)n\,, & \text{if } n \text{ is even,} \\ b_1\, \tilde{\mathrm{Le}}_{n-1}^{(a,b)} + b_2\, \tilde{\mathrm{Le}}_{n-2}^{(a,b)} + b_1 + (1 - b_2)n\,, & \text{if } n \text{ is odd.} \end{cases}$$

Certainly, by changing the initial conditions, we will obtain new sequences that will deserve study. We chose these because they seemed to be the best suited to the proposal in [6]. We can also aim for similar relations for the tri-periodic case as in [6]. We leave open to the interested reader a more in-depth study of these extensions.

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