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Note on Matuzsewska-Orlich indices and Zygmund inequalities

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Abstract

In this note we call attention to the fact that there exist some relations between the Matuszewska-Orlicz indices $m(\varphi)$ and $M(\varphi)$ of the function φ, and possible values of the constants in Zygmund type inequalities.

Key Words: Matuszewska-Orlicz indices, Zygmund type inequalities, almost monotonic functions
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1 Introduction

The main goal of this note is to call attention to the fact that there exist some relations between the Matuszewska-Orlicz indices $m(\varphi)$ and $M(\varphi)$ of the function φ, and possible values of the constants c_φ and C_φ in the inequalities

$$\int_0^h \frac{\varphi(t)}{t}dt \leq \frac{1}{c_\varphi}\varphi(h), \tag{1}$$

$$\int_h^\ell \frac{\varphi(t)}{t}dt \leq \frac{1}{C_\varphi}\varphi(h), \tag{2}$$

where $0 < h \leq \ell < \infty$, φ is a non-negative function, see Theorems 3.1 and 4.1.

Inequalities (1) and (2) are known as Zygmund type inequalities, we refer for instance to [1], where under some monotonicity conditions on φ there was shown in particular that Zygmund inequalities are equivalent to the so called Lozinsky and Bary-Stechkin conditions. In [2], [7] it was shown that monotonicity conditions on φ may be replaced by that of almost monotonicity, or more generally, by the condition $\varphi \in \widetilde{W}$, see Definition 2.1; recall that a non-negative function φ is called almost increasing if there exists a constant $c \geq 1$ such that $\varphi(x) \leq \varphi(y)$ for all $x \leq y$.

Note that we prefer to write constants on the right-hand sides of (1)-(2) as $\frac{1}{c}$ and $\frac{1}{C}$ by reasons which become clear in the sequel, see for instance Lemma 3.1 and inequality (5).

2 Preliminaries

The Matuszewska-Orlicz indices known in the theory of Orlicz spaces (see [5], [3] and [4], where they were studied mainly for Young functions φ), are defined as

$$m(\varphi) = \sup_{t>1} \frac{\ln \left[\underline{\lim}_{h\to 0} \frac{\varphi(th)}{\varphi(h)}\right]}{\ln\ t} \ = \lim_{t\to 0} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\varphi(th)}{\varphi(h)}\right]}{\ln\ t} \tag{1}$$

$$M(\varphi) = \inf_{t>1} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\varphi(th)}{\varphi(h)}\right]}{\ln\ t} \ = \lim_{t\to \infty} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\varphi(th)}{\varphi(h)}\right]}{\ln\ t}, \tag{2}$$

the definition being applicable to any non-negative-function φ, and

$$-\infty \leq m(\varphi) \leq M(\varphi) \leq +\infty$$

in this case.

Note that for $\varphi_\gamma(t) = t^\gamma \varphi(t)$ we have

$$m(\varphi_\gamma) = \gamma + m(\varphi) \ \text{ and } \ M(\varphi_\gamma) = \gamma + M(\varphi).$$

Definition 2.1. *By $W = W([0,\ell])$ we denote the class of non-negative almost increasing functions on $[0,\ell]$, positive on $(0,\ell)$ and by $\widetilde{W} = \widetilde{W}([0,\ell])$ we denote the class of functions on $[0,\ell]$, such that there exists an $a \in \mathbb{R}^1$ such that the function $x^a\varphi(x) \in W$.*

In the case $\varphi \in \widetilde{W}$, one has

$$-\infty < m(\varphi) \leq M(\varphi) \leq +\infty.$$

Various properties of the indices $m(\varphi)$ and $M(\varphi)$ were obtained in [3] and [4], and in [2], [7], [8], [9], [10], [11], [12] in connection with study of various operators in generalized Hölder spaces, where in particular it was shown that the validity of the Zygmund inequalities for a function $\varphi(t)$ may be characterized in terms of the indices $m(\varphi), M(\varphi)$.

In particular, the following property is known (for the proof see [2], Theorems 3.1 and 3.2 for $\varphi \in \widetilde{W}$, as stated in Theorem 2.1, and [3], Thm 6.4 or [4], Thm 11.8 under a different definition of the indices and other assumptions on φ)

Theorem 2.1. *Let $\varphi \in \widetilde{W}$. Then*

$$\int_0^h \frac{\varphi(t)}{t^{1+\gamma}} dt \le c \frac{\varphi(h)}{h^\gamma} \iff \gamma < m(\varphi), \tag{3}$$

$$\int_h^\ell \frac{\varphi(t)}{t^{1+\nu}} dt \le c \frac{\varphi(h)}{h^\nu} \iff \nu > M(\varphi). \tag{4}$$

3 A relation between the index $m(\varphi)$ and the constant c_φ

Given a non-negative function φ, let

$$I_-(\varphi) = \left\{ \gamma \in \mathbb{R}^1 : \text{ there exists } c = c(\varphi,\gamma) \text{ such that } \int_0^h \frac{\varphi(t)}{t^{1+\gamma}} dt \le \frac{1}{c} \frac{\varphi(h)}{h^\gamma} \right\}.$$

Obviously, if $\gamma \in I_-(\varphi)$, then $\gamma - a \in I_-(\varphi)$ for any $a > 0$, so that $I_-(\varphi)$ may be only an infinite interval starting from $-\infty$. For functions $\varphi \in \widetilde{W}$ it is known that the set $I_-(\varphi)$ is an open interval with the exactly calculated upper bound:

$$I_-(\varphi) = (-\infty, m(\varphi)), \tag{1}$$

which follows from (3).

In Lemma 3.1 we show that the fact itself that this interval is open, is valid for an arbitrary non-negative function φ, without any assumption on almost monotonicity of φ, and find a relation between the constants $c(\varphi,\gamma)$ and $c(\varphi,\gamma+\varepsilon)$.

Lemma 3.1. *Let $\varphi(t)$ be a non-negative function on $[0,\ell]$ such that the integral $\int_0^t \frac{\varphi(s)}{s} ds$ exists for every $t \in (0,\ell)$. If there holds inequality (1) with some $c_\varphi > 0$, then for any $\varepsilon \in (0, c_\varphi)$ there also holds the inequality*

$$\int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}} dt \le \frac{1}{c_\varphi - \varepsilon} \frac{\varphi(h)}{h^\varepsilon} \tag{2}$$

where c is the same as in (1).

Proof. Let

$$\Phi(t) = \int_0^t \frac{\varphi(s)}{s} ds.$$

The formula is valid

$$\int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}}dt = \frac{\Phi(h)}{h^\varepsilon} + \varepsilon \int_0^h \frac{\Phi(t)}{t^{1+\varepsilon}}dt. \tag{3}$$

Indeed,

$$\varepsilon \int_0^h \frac{\Phi(t)}{t^{1+\varepsilon}}dt = \varepsilon \int_0^h \frac{dt}{t^{1+\varepsilon}} \int_0^t \frac{\varphi(s)}{s}ds$$

$$= \varepsilon \int_0^h \frac{\varphi(s)}{s}ds \int_s^h \frac{dt}{t^{1+\varepsilon}} = \int_0^h \frac{\varphi(s)}{s}\left(\frac{1}{s^\varepsilon} - \frac{1}{h^\varepsilon}\right)ds$$

which yields (3).

Since $\Phi(h) \leq \frac{1}{c_\varphi}\varphi(h)$ by (1), from (3) we obtain

$$\int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}}dt \leq \frac{\varphi(h)}{c_\varphi h^\varepsilon} + \frac{\varepsilon}{c_\varphi} \int_0^h \frac{\varphi(t)}{t^{1+\varepsilon}}dt,$$

from which (2) follows. □

Corollary 3.1. *Let φ be a non-negative function on $[0, \ell]$ such that $\int\limits_0^\ell \frac{\varphi(t)}{t^{1+\gamma}}dt$ exists, $\gamma \in \mathbb{R}^1$. Then*

$$\int_0^h \frac{\varphi(t)}{t^{1+\gamma}}dt \leq \frac{1}{c_\gamma}\frac{\varphi(h)}{h^\gamma} \quad \Longrightarrow \quad \int_0^h \frac{\varphi(t)\,dt}{t^{1+\gamma+\varepsilon}} \leq \frac{1}{c_\gamma - \varepsilon}\,\frac{\varphi(h)}{h^{\gamma+\varepsilon}}$$

for any $\varepsilon < c_\gamma$.

Remark 3.1. *In case we pass from the factor $\frac{1}{t^\varepsilon}$ in (2) to a power of the logarithmic function, the corresponding statement becomes*

$$\int_0^h \frac{\varphi(t)}{t}dt \leq \frac{1}{c_\varphi}\varphi(h) \implies \int_0^h \frac{\varphi(t)\left(\ln \frac{h}{t}\right)^n}{t}dt \leq \frac{1}{c_\varphi^{n+1}n!}\varphi(h), \tag{4}$$

where $n = 1, 2, 3, ...$ which may be obtained by the successive application of the given inequality:

$$\varphi(h) \geq c_\varphi \int_0^h \frac{\varphi(t)}{t}\,dt \geq c_\varphi^2 \int_0^h \frac{dt}{t} \int_0^t \frac{\varphi(s)}{s}\,ds = c_\varphi^2 \int_0^h \frac{\varphi(s)\ln\frac{h}{s}}{s}\,ds \quad \textit{etc}$$

Theorem 3.1. *Let $\varphi \in \widetilde{W}$. If there holds inequality (1) with some constant $c_\varphi > 0$, then*

$$c_\varphi \leq m(\varphi). \tag{5}$$

Proof. Suppose to the contrary that $m(\varphi) < c_\varphi$. By Lemma 3.1, inequality (2) holds with every $\varepsilon \in (0, c_\varphi)$, in particular, with every $\varepsilon \in (\lambda, c_\varphi)$, $\lambda = \max\{m(\varphi), 0\}$, which is impossible, because for $\varphi \in \widetilde{W}$, inequality (2) implies $m(\varphi) > \varepsilon$ by 3. □

Corollary 3.2. *For the index $m(\varphi)$ of a function $\varphi \in W$ the estimate holds*

$$m(\varphi) \geq \inf_{t>0} \frac{\varphi(t)}{\Phi(t)} = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)}, \tag{6}$$

where $\Phi(t) = \int\limits_0^t \frac{\varphi(s)}{s} ds$.

Proof. Let $A = \sup_{h>0} \frac{\Phi(h)}{\varphi(h)}$. Let first $A = \infty$. Then the right-hand side of (6) is zero and also $m(\varphi) = 0$. Indeed, we have $m(\varphi) \geq 0$ for $\varphi \in W$ and in case $m(\varphi) > 0$ there holds (1) with a finite constant c_φ, which would mean that $A < \infty$. Therefore, (6) trivially holds in the case $A = \infty$.

Let $A < \infty$. Then (1) obviously holds with $c_\varphi = \frac{1}{A}$. Then $\frac{1}{A} \leq m(\varphi)$ by Lemma 3.1, which is inequality (6). □

Remark 3.2. *In case of power functions $\varphi(t) = t^\lambda$ we have*

$$m(\varphi) = M(\varphi) = \inf_{t>0} \frac{\varphi(t)}{\Phi(t)} = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} = \lambda,$$

but in the general case it may be that $m(\varphi) > \inf_{t>0} \frac{\varphi(t)}{\Phi(t)}$.

4 A relation between the index $M(\varphi)$ and the constant C_φ

Similarly to the previous section we reveal a relation between the upper index $M(\varphi)$ and the constant C_φ in the Zygmund inequality (2).

Let

$$I_+(\varphi) = \left\{ \gamma \in \mathbb{R}^1 : \text{ there exists } C = C(\varphi, \gamma) \text{ such that } \int\limits_h^l \frac{\varphi(t)}{t^{1+\gamma}} dt \leq \frac{1}{C} \frac{\varphi(h)}{h^\gamma} \right\}.$$

For functions $\varphi \in \widetilde{W}$ it is known that

$$I_+ = (M(\varphi), +\infty),$$

see (4). The following lemma exactifies the statement on the openness of the interval $(M(\varphi), +\infty)$ for an arbitrary non-negative function.

Lemma 4.1. *Let $\varphi(t)$ be a non-negative function on $[0,\ell]$ such that the integral $\int_t^\ell \frac{\varphi(s)}{s}ds$ exists for every $t \in (0,\ell)$. If there holds inequality* (2) *with some $C_\varphi > 0$, then for any $\varepsilon \in (0, C_\varphi)$ there also holds the inequality*

$$\int_h^\ell \frac{\varphi(t)}{t^{1-\varepsilon}}dt \le \frac{1}{C_\varphi - \varepsilon} h^\varepsilon \varphi(h) \tag{1}$$

where C_φ is the same as in (2).

Proof. Lemma 4.1 was proved in [6]. We give the proof here for the completeness of presentation. Let $\Phi_1(t) = \int_t^\ell \frac{\varphi(s)}{s}ds$. Similarly to (3) we have

$$\int_h^\ell \frac{\varphi(t)}{t^{1-\varepsilon}}dt = h^\varepsilon \Phi_1(h) + \varepsilon \int_h^\ell \frac{\Phi_1(t)}{t^{1-\varepsilon}}dt \tag{2}$$

by direct verification. Since $\Phi_1(h) \le \frac{1}{C_\varphi}\varphi(h)$ by (2), from (2) we obtain

$$C_\varphi \int_h^\ell \frac{\varphi(t)}{t^{1-\varepsilon}}dt \le h^\varepsilon \varphi(h) + \varepsilon \int_h^\ell \frac{\varphi(t)}{t^{1-\varepsilon}}dt,$$

from which (1) follows. □

Lemma 4.2. *Let $\varphi \in \widetilde{W}$. If there holds inequality* (2) *with some constant $C_\varphi > 0$, then*

$$M(\varphi) \le -C_\varphi.$$

Proof. Suppose to the contrary that $M(\varphi) > -C_\varphi$. By Lemma 4.1, inequality (1) holds with every $\varepsilon \in (0, C_\varphi)$, in particular, with every $\varepsilon \in (\mu, C_\varphi),\ \mu = \max\{-M(\varphi), 0\}$, which is impossible, because for $\varphi \in \widetilde{W}$, inequality (1) implies $M(\varphi) < -\varepsilon$ by (4). □

Theorem 4.1. *If a function $\varphi \in \widetilde{W}$ admits estimate* (2) *with some constant $C_\varphi > 0$, then for the index $M(\varphi)$ the estimate holds*

$$M(\varphi) \le -\inf_{0<t\le\ell} \frac{\varphi(t)}{\Phi_1(t)} = \sup_{0<t\le\ell} \frac{t\Phi_1'(t)}{\Phi_1(t)}, \tag{3}$$

where $\Phi_1(t) = \int_t^\ell \frac{\varphi(s)}{s}ds$.

Proof. Let $A_1 = \sup_{0<t<l} \frac{\Phi(t)}{\varphi(t)}$. Inequality (2) obviously holds with $C_\varphi = \frac{1}{A_1}$. Then $\frac{1}{A_1} \le -M(\varphi)$ by Lemma 4.2, which is inequality (3). □

Remark 4.1. *The indices*

$$p(\varphi) = \inf_{0<x\le\ell} \frac{x\varphi'(x)}{\varphi(x)}, \quad q(\varphi) = \sup_{0<x\le\ell} \frac{x\varphi'(x)}{\varphi(x)} \tag{4}$$

are known as Simonenko indices, see [13], and it is known that

$$p(\varphi) \le m(\varphi) \le M(\varphi) \le q(\varphi), \tag{5}$$

see [4], Theorem 11.11. In these terms, inequalities (6) *and* (3)*, in case* $\varphi \in W$, *mean that*

$$p(\Phi) \le m(\varphi) \le M(\varphi) \le q(\Phi_1). \tag{6}$$

Observe that although we can write, for instance,

$$p(\Phi) \le m(\Phi) \le M(\Phi) \le q(\Phi),$$

to derive the left-hand side inequality $p(\Phi) \le m(\varphi)$ *in* (6) *from here, we would like to have the property* $m(\Phi) = m(\varphi)$, *which is true in the case* $0 < m(\varphi) \le M(\varphi) < \infty$ *because* $\Phi \sim \varphi$ *in this case and then the functions* Φ *and* φ *have coinciding indices, see [4], Theorem 11.4. Similarly one has* $M(\Phi_1) = M(\varphi)$ *when* $-\infty < m(\varphi) \le M(\varphi) < 0$.

5 A generalization of Lemmas 3.1 and 4.1

Based on the passage from (1) to (2) and the example given in (4), we now consider a possibility to trace a similar passage when one deals with the scale of functions more fine than just the scale of power (or power-logarithmic) functions.

In the sequel the notation $AC(0,\ell)$ stands for the set of functions on $(0,\ell)$ absolutely continuous on every closed subinterval of $(0,\ell)$.

Lemma 5.1. *Suppose that*

$$\int_0^h \frac{\varphi(t)}{t} dt \le \frac{1}{c_0}\varphi(h) \tag{1}$$

for some $c_0 > 0$. *Then a similar inequality*

$$\int_0^h \frac{\varphi(t)}{t\nu(t)} dt \le \frac{1}{c_0 - \delta} \frac{\varphi(h)}{\nu(h)} \tag{2}$$

holds, where $\nu(t)$ *is any non-negative function on* $[0,\ell]$ *such that* $\frac{1}{\nu} \in AC(0,\ell)$, *and*

$$\delta =: \sup_{t\in[0,\ell]} \frac{t|\nu'(t)|}{\nu(t)} < c_0. \tag{3}$$

Proof. Integration by parts yields

$$\int_0^h \frac{\varphi(t)\,dt}{t\nu(t)} = \frac{\Phi(h)}{\nu(h)} + \int_0^h \frac{\nu'(t)}{\nu^2(t)}\Phi(t)dt \tag{4}$$

since $\lim_{h\to 0}\frac{\Phi(h)}{\nu(h)} = 0$. To check the latter, in view of (1) it suffices to show that $\lim_{h\to 0}\frac{\varphi(h)}{\nu(h)} = 0$, for which it is sufficient to verify that $m\left(\frac{\varphi}{\nu}\right) > 0$. Since $m\left(\frac{\varphi}{\nu}\right) \geq m(\varphi) + m\left(\frac{1}{\nu}\right) = m(\varphi) - M(\nu)$, we then may only check that $M(\nu) < m(\varphi)$. The latter follows from condition (3), which implies that $M(\nu) \leq q(\nu) < c_0(\leq m(\varphi))$.

From (4), by assumption (1) we obtain

$$\int_0^h \frac{\varphi(t)\,dt}{t\nu(t)} \leq \frac{1}{c_0}\left[\frac{\varphi(h)}{\nu(h)} + \int_0^h \frac{|\nu'(t)|}{\nu^2(t)}\varphi(t)dt\right] \tag{5}$$

or

$$\int_0^h \left(1 - \frac{1}{c_0}\frac{t|\nu'(t)|}{\nu(t)|}\right)\frac{\varphi(t)}{t\nu(t)}dt \leq \frac{1}{c_0}\frac{\varphi(h)}{\nu(h)}. \tag{6}$$

By assumption (3) we have $1 - \frac{1}{c_0}\frac{t|\nu'(t)|}{\nu(t)|} \geq 1 - \frac{\delta}{c_0}$ which yields (2). □

Lemma 5.2. *Suppose that*

$$\int_h^\ell \frac{\varphi(t)}{t}dt \leq \frac{\varphi(h)}{C_0} \tag{7}$$

for some $C_0 > 0$*. Then a similar inequality*

$$\int_h^\ell \frac{\varphi(t)\lambda(t)}{t}dt \leq \frac{\lambda(h)\varphi(h)}{C_0 - \delta} \tag{8}$$

holds, where $\lambda(t)$ *is any non-negative function in* $AC(0,\ell)$*, and*

$$\delta =: \sup_{t\in[0,\ell]} \frac{t|\lambda'(t)|}{\lambda(t)} < C_0. \tag{9}$$

Proof. Integrating by parts, we obtain

$$\int_h^\ell \lambda(t)\frac{\varphi(t)}{t}dt = \lambda(h)\Phi_1(h) + \int_h^\ell \lambda'(t)\Phi_1(t)dt, \quad \Phi_1(t) = \int_h^\ell \frac{\varphi(t)}{t}\,dt. \tag{10}$$

By assumption (7) we then have

$$\int_h^\ell \frac{\lambda(t)\varphi(t)\,dt}{t} \leq \frac{1}{C_0}\left[\lambda(h)\varphi(h) + \int_h^\ell |\lambda'(t)|\varphi(t)dt\right] \tag{11}$$

or

$$\int_h^\ell \left(1 - \frac{1}{C_0}\frac{t|\lambda'(t)|}{\lambda(t)|}\right)\frac{\lambda(t)\varphi(t)}{t}dt \leq \frac{1}{C_0}\lambda(h)\varphi(h), \tag{12}$$

which yields (8) by (9). □

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