Armenian Journal of Mathematics
Volume 4, Number 1, 2012, 25–43

On Analytic Continuation of the Power Series Outside of the Convergence Disc

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Abstract

In current work we discuss the issue of the analytic continuation of a power series along a logarithmic spiral outside of the convergence disc. A necessary and sufficient condition in terms of the interpolating entire function is obtained. Moreover, the relation in between the possibility of analytic continuation and the density of the lacunas of a power series is studied.

Key Words: power series, analytic continuation, interpolating entire function, lacunas, density.
Mathematics Subject Classification 2000: 30B10, 30B30, 30B40.

1 Introduction

In the theory of analytic continuation of analytic functions the notion of an *analytic element* (shortly: *element* or *power series, series*) plays a central role. An analytic element with a center $a \in \mathbb{C}$ is the power series

$$f(z) = \sum_{n=0}^{\infty} f_n (z-a)^n, \tag{1}$$

which converges in some open disc $D_r(a) = \{w \in \mathbb{C} \mid |w-a| < r\}$ which is called the *convergence disc*. An analytic element with center $a = \infty$ is a series of the form (1), where the term $(z-a)$ is replaced by a term z^{-1}. An element with center $z = \infty$ converges in a complement of some closed disc i.e. $D_r(\infty) = \overline{\mathbb{C}} \setminus \overline{D_r(0)}$. Denote by H_a the set of elements with center $a \in \overline{\mathbb{C}}$. An element is called *normalized* if it has a center 0 and radius of convergence equal to 1, i.e. an element of the form

$$f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad \limsup_{n\to\infty} |f_n|^{1/n} = 1. \tag{2}$$

An element (f_1, D_1) (i.e. an element f_1, which has a convergence disc D_1) is called analytic continuation of the element (f_0, D_0) if $D_0 \cap D_1 \neq \emptyset$ and $f_0 = f_1$ on the intersection $D_0 \cap D_1$. If so, we write $(f_0, D_0) \sim (f_1, D_1)$. If we have a sequence of analytic elements (an *analytic chain*)

$$(f_0, D_0) \sim (f_1, D_1) \sim \cdots \sim (f_n, D_n), \tag{3}$$

then (f_n, D_n) is called analytic continuation of the element (f_0, D_0) *along a chain*. Some of the important issues of theory of analytic continuations are the following:

- Find a criterion for an element $f \in H_a$, which ensures that f can be analytically continued to a domain $\Omega \subset \overline{\mathbb{C}}$, which contains a. Or more generally, characterize the analyticity domain of the element f.
- Recover the analytic continuation of the element $f \in H_a$ in the domain $\Omega \subset \overline{\mathbb{C}}$, already knowing that the continuation exists.
- Localize the singularities of the element $f \in H_a$ on the boundary of the convergence disc and outside of it.

These problems can be considered as subparts of a more global one: *characterize global properties of a complete analytic function using only its local data.*

Let us mention some classical results in this direction, which can be found in books by Dienes [6] and Tsuji [8].

1. **L. Kronecker's** criterion for the element (1) to be a rational function.
2. **J. Hadamard's** criterion for the element (2) to be a meromorphic function.
3. **G. Eisenstein** obtained a necessary condition on the coefficients of the element (1) for f being an algebraic function.
4. **I. Schur's** theorem on boundedness of the element (2) on the unit disc.
5. **L. de Branges'** proof of the *Bieberbach conjecture*, which is a necessary condition for the element (2) to be a one-to-one map on the unit disc.

In current work we consider the possibility of analytic continuation of a normalized power series along a segment $\gamma_\rho,\ \rho > 1$ (see (9)) of an *logarithmic α-spiral* outside of the convergence domain. We obtain a criterion that is a generalization of a result by **N. Arakelian** [3], where he considers the case $\alpha = 0$, i.e. when analytic element (2) is continued along a segment $(-\rho, -1]$.

We also discuss the localization of the singularities of an element of a form (2). More precisely: is it possible to find the first singularity of the element (2) along an α-spiral in terms of its coefficients? In other words we inquire if it is possible to find the biggest $\rho > 1$ in terms of $\{f_n\}$ such that the series (2) continue analytically along γ_ρ.

We give a partial answer to this question by showing that one of the classical tools to characterize the singularities of an analytic element: the density of the zero set of its coefficients (or so called *lacunas*), do not give an answer in this case. To do this, we fix any density $0 \leq \Delta < 1$ and any number $\rho > 1$ and construct an element (2) that has lacunas of density Δ and continues analytically along γ_ρ.

At the end we discuss the balance in between the growth of an entire function at the infinity and the density of its natural zeros (zeros that are natural numbers).

More precisely, in Theorem 1 we obtain that an analytic element (2) has an analytic continuation along γ_ρ, $\rho > 1$ if and only if there exists an entire function φ that interpolates the coefficients of (2) and satisfies the growth conditions (11), (12). Hence, natural zeros of φ are the lacunas of the element (2). Motivated by this, we discuss the density of natural zeros of an entire function that has the growth (11), (12) and prove that such a function cannot have zeros of density 1.

This can be seen as a complementary statement to the one asserting existence of elements with any given density $0 \leq \Delta < 1$, which can be continued analytically along γ_ρ for some $\rho > 1$.

In what follows we introduce the notation used in the work.

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\},\ \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\},$$

$$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}, \qquad \mathbb{R}^- = \{x \in \mathbb{R} :\ x \leq 0\}.$$

For a domain $\Omega \in \overline{\mathbb{C}}$ the set $H(\Omega)$ is the set of holomorphic functions on Ω. For a set $E \subset \mathbb{C}$ denote by

E^0 the interior, $\overline{E}$ the closure and by ∂E the boundary of E,

$E^* = \{\overline{z} \mid z \in E\}$ the conjugate set of E,

$E^{-1} = \{z^{-1} \mid z \in E\}$, if $0 \overline{\in} E$.

For $E_1, E_2 \subset \mathbb{C}$ and $z \in \mathbb{C}$

$$E_1 + E_2 = \{\zeta_1 + \zeta_2 \mid \zeta_1 \in E_1,\ \zeta_2 \in E_2\},$$

$$E_1E_2 = \{\zeta_1\zeta_2 \mid \zeta_1 \in E_1,\ \zeta_2 \in E_2\},$$

$$E + z = E + \{z\}, \quad zE = \{z\}E.$$

Furthermore,

$\Delta(\alpha, \beta) = \{re^{i\theta} \mid r \geq 0, \quad \theta \in [\alpha, \beta]\}$ - angular sector,

$\Pi = \Delta(-\frac{\pi}{2}, \frac{\pi}{2}) = \mathbb{R}^+ \times \mathbb{R}$ - right halfplane

$\Pi_\alpha = e^{i\alpha}\Pi, \quad \alpha \in \mathbb{R}$ - closed halfplane.

Moreover,

1. For a set $E \subset \mathbb{C}$ define its *support function*

$$K_E(t) = \sup_{\zeta \in E}\{\Re(\zeta e^{-it})\}, \qquad t \in \mathbb{R}. \tag{4}$$

2. Let $\Delta \subset \overline{\mathbb{C}}$ be an angular sector. Then for a function $\varphi \in H(\Delta)$ define its *exponential type* as

$$\sigma_\varphi = \limsup_{z \to \infty} \frac{\log |\varphi(z)|}{|z|}. \tag{5}$$

3. For a function $\varphi \in H(\Delta)$ its *inner exponential type* is the number

$$\sigma_{\varphi,\Delta} = \sup_{\Lambda \subset \Delta^0 \cup \{0\}} \sigma(\Lambda). \tag{6}$$

If $\sigma_{\varphi,\Delta} < \infty$ we write $\varphi \in B(\Delta)$.

For a sector $\Delta = \Delta(\alpha, \beta)$ the function

$$h_\varphi(\theta) = \limsup_{r \to \infty} \frac{\log |\varphi(re^{i\theta})|}{r}, \qquad \theta \in (\alpha, \beta), \tag{7}$$

is called the *indicator function* of φ. The set

$$I_\varphi = \bigcup_{\theta \in (\alpha,\beta)} \{\zeta \in \mathbb{C} \mid \Re(\zeta e^{-i\theta}) \leq h_\varphi(\theta)\} \tag{8}$$

is called the *indicator diagram* of φ.

2 Analytic continuation criterion

Here we state and prove a criterion for an element of a form (2) to be analytically continued along a logarithmic α-spiral.

For $\alpha \in \mathbb{R}$ let $L_\alpha = \{0\} \cup \{t^{1+i\alpha} = \exp(i\alpha \log t) | \ t \in (0, \infty)\}$ be the logarithmic α-spiral. For any $a \in \mathbb{C}$ the set aL_α is a α-spiral passing through the point a.

Consider a normalized element (2) and a "segment"

$$\gamma_\rho = \{-t^{1+i\alpha} = \exp(i\alpha \log t + i\pi) \mid t \in [1, \rho)\}, \quad \rho > 1, \tag{9}$$

of a α-spiral passing through the point -1. Then, the following result holds:

Theorem 1. *For the analytic element* (2) *to be continued along γ_ρ, for some $\rho > 1$, it is necessary and sufficient that there exists a function $\varphi \in B(\Pi_{-\beta})$ such that*

$$\varphi(n) = f_n, \quad \forall n \in \mathbb{N}_0, \tag{10}$$

and

$$\liminf_{\theta \to \pi/2 - \beta} \frac{\pi \cos\beta - h_\varphi(\theta)}{\pi/2 - |\theta + \beta|} \geq \pi \sin\beta + \frac{\log\rho}{\cos\beta}, \tag{11}$$

$$\liminf_{\theta \to -\pi/2 - \beta} \frac{\pi \cos\beta - h_\varphi(\theta)}{\pi/2 - |\theta + \beta|} \geq -\pi \sin\beta + \frac{\log\rho}{\cos\beta}, \tag{12}$$

where $\beta = \arctan\alpha$.

To prove this theorem we need a general criterion on analytic continuation of an element.

A compact set $E \subset \overline{\mathbb{C}}$ is called *logarithmic convex* (or *log-convex*) if

$$E = \exp(L) \cup \{0, \infty\}, \tag{13}$$

for some $L \subset \mathbb{C}$ closed, convex set. If L is the smallest set with these properties, then $L = L(E)$ is called the *logarithmic diagram* of E.

Note that $L(E)$ does not contain vertical segments of length greater than 2π and if $0 \in \partial E$ or $\infty \in \partial E$, then it does not contain vertical segments of length 2π.

The set $L(E)$ is bounded if and only if $0, \infty \overline{\in} E$. Also, note that for every $m \in \mathbb{Z}$, $L(E) + 2\pi m i$ is also a logarithmic diagram for E. If $0 \in \partial E$ or $\infty \in \partial E$, these are all possible logarithmic diagrams of E.

Furthermore, if $0 \in E$, then $L(E)$ is unbounded to the left and there exists an angle $\beta \in (-\pi/2, \pi/2)$, such that $L(E)$ contains parallel rays, which form an angle β with the semiaxis $(-\infty, 0]$. We call this angle the *direction* of the set $L(E)$ and denote by $\beta = \beta(E)$. If $\infty \in E$ and $0 \overline{\in} E$, then $\beta(E) = \beta(E^{-1})$.

Now let us formulate the aforementioned general result on analytic continuation of a power series.

Theorem 2 ([1]). *Let E be a log-convex compact subset of $\mathbb{C}$ such that $0 \in E$ and the set $\overline{\mathbb{C}} \setminus E$ is connected. Furthermore, let $L(E)$ be the logarithmic diagram of E and β be its direction. Then the power series*

$$\sum_{n=0}^{\infty} z^{-n-1} f_n, \qquad n \in N_0 \tag{14}$$

converges in the neighborhood of the ∞ *and defines a holomorphic function* $f \in H(\overline{C} \setminus E)$ *if and only if there exists a function* $\varphi \in H(\mathbb{C})$ *of order* ≤ 1 *and of finite exponential type on* $\Pi_{-\beta}$ *(on* $\overline{\mathbb{C}}$*, if* $0 \in E^0$*) such that*

$$\varphi(n) = f_n, \qquad \forall n \in N_0. \tag{15}$$

Additionally, $I_\varphi \subset L^*(E)$ *or equivalently*

$$h_\varphi(\theta) \leq K_{L(E)}(-\theta), \qquad |\theta + \beta| < \frac{\pi}{2}. \tag{16}$$

Remark 1. *a) In the sufficiency part of the Theorem 2 one can relax the conditions on* φ *by just requiring* $\varphi \in B(\Pi_{-\beta})$ *and that equations* (10), (16) *hold.*

b) To reformulate Theorem 2 for the elements with center $a = 0$*, assume that* E *is a log-convex set in* $\overline{\mathbb{C}} \setminus \{0\}$ *with connected complement. Then, to obtain necessary and sufficient conditions for* f *to be holomorphic in* $\overline{\mathbb{C}} \setminus E$*, one needs to replace* $L(E)$ *by* $-L(E) = L(E^{-1})$ *in the statement of the theorem, i.e.*

$$I_\varphi \subset -L^*(E). \tag{17}$$

Now let us prove the Theorem 1.

Proof of the Theorem 1. Without loss of generality we can assume that $\alpha \geq 0$.

Necessity. Suppose that element (2) has an analytic continuation along γ_ρ. This means that $f \in H(G)$, for some domain $G \supset D_1(0) \cup \gamma_\rho$. Then, it is not difficult to see that there exists a function $\varepsilon : [1, \rho] \to \mathbb{R}^+$ such that

a) $\varepsilon \in C([1, \rho])$,

b) ε is strictly decreasing on $[1, \rho]$,

c) $\varepsilon(t) > 0, \qquad t \in [1, \rho)$,

d) $\varepsilon(\rho) = 0$,

e) $D_1(0) \cup \{t \exp i(\alpha \log t + \pi + \theta) \, || \theta| \leq \varepsilon(t), \ t \in [1, \rho)\} \subset G$.

Now consider the function

$$\gamma(t) = \frac{2}{(\rho - 1)^2} \int_t^\rho \varepsilon(\tau) \log \frac{\tau}{t} d\tau, \qquad t \in [1, \rho]. \tag{18}$$

Lemma 1. *The function* γ *defined by the formula* (18) *has following properties*

a) $\gamma \in C^2([1, \rho])$,

b) $0 < \gamma(t) \leq \varepsilon(t), \qquad t \in [1, \rho)$, *c)* $\gamma(\rho) = \gamma'(\rho) = \gamma''(\rho) = 0$,

d) $t\gamma'(t)$ *is strictly increasing on the interval* $[1, \rho]$.

Proof of the Lemma 1. a) From (18) we have that

$$\gamma'(t) = -\frac{2}{t(\rho-1)^2}\int_t^\rho \varepsilon(\tau)d\tau \tag{19}$$

and

$$\gamma''(t) = \frac{2}{t^2(\rho-1)^2}\int_t^\rho \varepsilon(\tau)d\tau + \frac{2\varepsilon(t)}{t(\rho-1)^2}, \tag{20}$$

therefore, since $\varepsilon \in C([1, \rho])$ we deduce that $\gamma \in C^2([1, \rho])$.

b) Note that the integrand in the equation (18) is positive, therefore $\gamma(t) > 0$. Now let us show that $\gamma(t) \leq \varepsilon(t)$. The function ε is strictly decreasing, hence

$$\begin{aligned}
\gamma(t) = \frac{2}{(\rho-1)^2}\int_t^\rho \varepsilon(\tau)\log\frac{\tau}{t}d\tau &\leq \frac{2\varepsilon(t)}{(\rho-1)^2}\int_t^\rho \log\frac{\tau}{t}d\tau \\
&\leq \frac{2\varepsilon(t)}{(\rho-1)^2}\int_t^\rho \left(\frac{\tau}{t}-1\right)d\tau \\
&= \frac{2\varepsilon(t)}{(\rho-1)^2}\frac{(\rho-t)^2}{2t} \\
&\leq \frac{2\varepsilon(t)}{(\rho-1)^2}\frac{(\rho-1)^2}{2} = \varepsilon(t).
\end{aligned}$$

c) To prove this item just plug in $t = \rho$ in the formulas obtained in the item *a*).

d) We have that

$$(t\gamma'(t))' = \gamma'(t) + t\gamma''(t) = \frac{2\varepsilon(t)}{(\rho-1)^2} > 0, \quad t \in [1, \rho)$$

therefore, the function $t\gamma'(t)$ is strictly increasing.

The lemma is proven. □

Consider following set

$$E = \overline{\mathbb{C}} \setminus \left(D_1(0) \cup \{t \exp i(\alpha \log t + \pi + \theta) \mid |\theta| < \gamma(t),\ t \in [1, \rho)\}\right). \tag{21}$$

Note that $E \subset \overline{\mathbb{C}} \setminus \{0\}$ is a compact set. Furthermore, by the Lemma 1 we have that $\gamma(t) \leq \varepsilon(t)$, hence

$$\begin{aligned}
\overline{\mathbb{C}} \setminus E &= D_1(0) \cup \{t \exp i(\alpha \log t + \pi + \theta) \mid |\theta| < \gamma(t),\ t \in [1, \rho)\} \\
&\subset D_1(0) \cup \{t \exp i(\alpha \log t + \pi + \theta) \mid |\theta| \leq \varepsilon(t),\ t \in [1, \rho)\}.
\end{aligned}$$

On the other hand

$$D_1(0) \cup \{t \exp i(\alpha \log t + \pi + \theta) \mid |\theta| \leq \varepsilon(t),\ t \in [1, \rho)\} \subset G,$$

thus $\overline{\mathbb{C}} \setminus E \subset G$, therefore $f \in H(\overline{\mathbb{C}} \setminus E)$.

Let us now check that E is a log-convex set. Introduce a function

$$\begin{cases} \widetilde{\gamma}(t) = \gamma(e^t), \quad t \in [0, \log \rho], \\ \widetilde{\gamma}(t) = 0, \quad t \in (\log \rho, \infty). \end{cases} \tag{22}$$

Lemma 2. *The function $\widetilde{\gamma}$ has following properties*

a) $\widetilde{\gamma} \in C^2(\mathbb{R}^+)$,

b) $\widetilde{\gamma}$ *is convex.*

Proof of the Lemma 2. From the items $a)$ and $c)$ of the Lemma 1 we have that $\widetilde{\gamma} \in C^2(\mathbb{R}^+)$.

Furthermore,

$$\begin{cases} \widetilde{\gamma}'(t) = e^t \gamma(e^t), \quad t \in [0, \log \rho), \\ \widetilde{\gamma}'(t) = 0, \quad t \in [\log \rho, \infty), \end{cases} \tag{23}$$

therefore, from the properties $c)$ and $d)$ of γ (see Lemma 1) we obtain that $\widetilde{\gamma}'(t)$ is a nondecreasing function, hence $\widetilde{\gamma}$ is convex. □

Remark 2. *From the property $c)$ of the function γ (Lemma 1) we have that $\widetilde{\gamma}'$ is strictly increasing on the interval $[0, \log \rho]$.*

Consider following functions

$$\begin{cases} z_1(t) = t + i(\alpha t + \pi - \widetilde{\gamma}), \\ z_2(t) = t + i(\alpha t - \pi + \widetilde{\gamma}), \quad t \in \mathbb{R}^+. \end{cases} \tag{24}$$

Furthermore, let

$$\gamma_1 = z_1(\mathbb{R}^+),\ \gamma_2 = z_2(\mathbb{R}^+),\ I = \{\zeta \in \mathbb{C} \mid \Re\zeta = 0,\ |\Im\zeta| \le \pi - \widetilde{\gamma}(0)\}. \tag{25}$$

Now let L be the following set

$$L = \{t + is \mid \alpha t - \pi + \widetilde{\gamma}(t) \le s \le \alpha t + \pi - \widetilde{\gamma}(t),\ t \in \mathbb{R}^+\}. \tag{26}$$

Note that $L = L(E)$ is the logarithmic diagram of the set E. Furthermore, the direction of $L(E)$ is $\beta(E) = \arctan \alpha$.

Now applying Theorem 2 to the function $f \in H(\overline{\mathbb{C}} \setminus E)$ and to the set E, we obtain that there exists a function $\varphi \in H(\mathbb{C})$ of an order ≤ 1 and of a finite exponential type on the halfplane $\Pi_{-\beta}$ such that

$$\varphi(n) = f_n, \quad n \in \mathbb{N}_0 \tag{27}$$

and

$$h_\varphi(\theta) \le K_{-L(E)}(-\theta), \quad |\theta + \beta| < \frac{\pi}{2}. \tag{28}$$

Consequently, to finish the proof of the necessity part of the Theorem 1, it suffices to show that

$$\liminf_{\theta\to\pi/2-\beta}\frac{\pi\cos\beta-K_{-L(E)}(-\theta)}{\pi/2-|\theta+\beta|}\geq\pi\sin\beta+\frac{\log\rho}{\cos\beta}\tag{29}$$

and

$$\liminf_{\theta\to-\pi/2-\beta}\frac{\pi\cos\beta-K_{-L(E)}(-\theta)}{\pi/2-|\theta+\beta|}\geq-\pi\sin\beta+\frac{\log\rho}{\cos\beta}.\tag{30}$$

We prove only the first inequality as the second can be proven in a same way.

Note that

$$K_{-L(E)}(-\theta)=K_{L(E)}(\pi-\theta),\quad K_{L(E)}(\pi-(\pi/2-\beta))=\pi\cos\beta.\tag{31}$$

Furthermore, if $0<\pi/2-\beta-\theta<\eta$, for sufficiently small η, then

$$K_{L(E)}(\pi-\theta)=\sup_{t\in[0,\log\rho]}\Re(z_1(t)e^{i(\theta-\pi)})=\max_{t\in[0,\log\rho]}(-t\cos\theta+(\alpha t+\pi-\widetilde{\gamma})\sin\theta).\tag{32}$$

But

$$\max_{t\in[0,\log\rho]}(-t\cos\theta+(\alpha t+\pi-\widetilde{\gamma})\sin\theta)=-t_\theta\cos\theta+(\alpha t_\theta+\pi-\widetilde{\gamma}_\theta)\sin\theta,\tag{33}$$

where t_θ is the only solution of the equation

$$(-t\cos\theta+(\alpha t+\pi-\widetilde{\gamma})\sin\theta)'=-\cos\theta+\alpha\sin\theta-\widetilde{\gamma}'(t)\sin\theta=0.$$

So we have that

$$\widetilde{\gamma}'(t_\theta)=-\cot\theta+\alpha.\tag{34}$$

Since the function $\widetilde{\gamma}'$ is continuous, invertible and

$$\lim_{\theta\to\pi/2-\beta}\widetilde{\gamma}'(t_\theta)=\lim_{\theta\to\pi/2-\beta}(-\cot\theta+\alpha)=0=\widetilde{\gamma}(\log\rho),$$

we obtain that

$$\lim_{\theta\to\pi/2-\beta}t_\theta=\log\rho.\tag{35}$$

Summarizing the inequalities above, we obtain

$$\begin{aligned}\liminf_{\theta\to\frac{\pi}{2}-\beta}\frac{\pi\cos\beta-K_{-L(E)}(-\theta)}{\frac{\pi}{2}-\beta-\theta}&=\liminf_{\theta\to\frac{\pi}{2}-\beta}\frac{\pi\cos\beta-(-t_\theta\cos\theta+(\alpha t_\theta+\pi-\widetilde{\gamma}(t_\theta))\sin\theta)}{\frac{\pi}{2}-\beta-\theta}\\ =\lim_{\theta\to\frac{\pi}{2}-\beta}\frac{\pi\sin(\frac{\pi}{2}-\beta)-\pi\sin\theta}{\frac{\pi}{2}-\beta-\theta}&-\lim_{\theta\to\frac{\pi}{2}-\beta}t_\theta\sin\theta\frac{\cot(\frac{\pi}{2}-\beta)-\cot\theta}{\frac{\pi}{2}-\beta-\theta}+\lim_{\theta\to\frac{\pi}{2}-\beta}\frac{\widetilde{\gamma}(t_\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta}\\ &=\pi\sin\beta+\frac{\log\rho}{\cos\beta}.\end{aligned}$$

So it remains to show that

$$\lim_{\theta\to\frac{\pi}{2}-\beta}\frac{\widetilde{\gamma}(t_\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta}=0.$$

Since $\widetilde{\gamma}$ is convex, then

$$\widetilde{\gamma}(\log\rho) \geq \widetilde{\gamma}(t_\theta) + (\log\rho - t_\theta)\widetilde{\gamma}'(t_\theta),$$

hence

$$0 < \frac{\widetilde{\gamma}(t_\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta} \leq \frac{(t_\theta - \log\rho)\widetilde{\gamma}'(t_\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta} = \frac{(t_\theta - \log\rho)(\cot(\frac{\pi}{2}-\beta) - \cot\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta},$$

but

$$\lim_{\theta\to\frac{\pi}{2}-\beta} \frac{(t_\theta - \log\rho)(\cot(\frac{\pi}{2}-\beta) - \cot\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta} = 0,$$

therefore

$$\lim_{\theta\to\frac{\pi}{2}-\beta} \frac{\widetilde{\gamma}(t_\theta)\sin\theta}{\frac{\pi}{2}-\beta-\theta} = 0.$$

This finishes the necessity part of the proof of the Theorem 1.

Sufficiency. So we have a function $\varphi \in B(\Pi_{-\beta})$, such that equations (10)-(12) hold. By **V. Bernstein's** theorem (see [5]) we have that $0 \in I_\varphi$ or

$$h_\varphi(0) = \limsup_{n\to\infty} \log|\varphi(n)|^{1/n} = 0. \tag{36}$$

Now let us show that equations (11),(12) and (36) yield that

$$h_\varphi(\theta) \leq \pi|\sin\theta|, \quad \theta \in (-\pi/2-\beta, \pi/2-\beta). \tag{37}$$

Since $h_\varphi(\theta)$ is trigonometrically convex (see [4]), we have that

$$h_\varphi(\theta)\sin\widetilde{\theta} \leq h_\varphi(0)\sin(\widetilde{\theta}-\theta) + h_\varphi(\widetilde{\theta})\sin\theta, \quad 0 \leq \theta < \widetilde{\theta} < \frac{\pi}{2}-\beta,$$

hence using (36), we obtain

$$h_\varphi(\theta) \leq \frac{h_\varphi(\widetilde{\theta})\sin\theta}{\sin\widetilde{\theta}}, \quad 0 \leq \theta < \widetilde{\theta} < \frac{\pi}{2}-\beta.$$

Now using the inequality (11), we get

$$h_\varphi(\theta) \leq \limsup_{\widetilde{\theta}\to\frac{\pi}{2}-\beta} \frac{h_\varphi(\widetilde{\theta})\sin\theta}{\sin\widetilde{\theta}} \leq \frac{\pi\cos\beta\sin\theta}{\sin(\frac{\pi}{2}-\beta)} = \pi\sin\theta, \quad 0 \leq \theta < \frac{\pi}{2}-\beta.$$

In a same way one proves the inequality (37) for the case $\pi/2-\beta < \theta < 0$.

From (37) it follows that

$$I_\varphi \subset \bigcap_{\theta\in(-\frac{\pi}{2}-\beta,\frac{\pi}{2}-\beta)} \{\zeta : \Re(\zeta e^{-i\theta}) \leq \pi|\sin\theta|\}.$$

On the other hand

$$\bigcap_{\theta\in(-\frac{\pi}{2}-\beta,\frac{\pi}{2}-\beta)} \{\zeta : \Re(\zeta e^{-i\theta}) \leq \pi|\sin\theta|\} =$$

$$\{\zeta : \Re\zeta \le 0\} \cap \{\zeta : \Im\zeta + \alpha\Re\zeta \le \pi\} \cap \{\zeta : \Im\zeta + \alpha\Re\zeta \ge -\pi\},$$

thus

$$I_\varphi \subset \{\zeta : \Re\zeta \le 0\} \cap \{\zeta : \Im\zeta + \alpha\Re\zeta \le \pi\} \cap \{\zeta : \Im\zeta + \alpha\Re\zeta \ge -\pi\}.$$

Let us show that

$$-t + i(\pi + \alpha t), -t + i(-\pi + \alpha t)\overline{\in} I_\varphi, \quad t \in [0, \log\rho). \tag{38}$$

Suppose that this is not true, i.e. there exists $t_0 \in [0, \log\rho)$ such that

$$z_0 = -t_0 + i(\pi + \alpha t_0) \in I_\varphi. \tag{39}$$

Since I_φ is convex (see [4]), then $I_0 = [0, z_0] \subset I_\varphi$, therefore

$$h_\varphi(\theta) \ge K_{I_0}(\theta), \quad \theta \in (-\frac{\pi}{2} - \beta, \frac{\pi}{2} - \beta),$$

from where we have that

$$\liminf_{\theta\to\frac{\pi}{2}-\beta} \frac{\pi\cos\beta - K_{I_0}(\theta)}{\frac{\pi}{2} - |\theta+\beta|} \ge \liminf_{\theta\to\frac{\pi}{2}-\beta} \frac{\pi\cos\beta - h_\varphi(\theta)}{\frac{\pi}{2} - |\theta+\beta|} \ge \pi\sin\beta + \frac{\log\rho}{\cos\beta}.$$

Note that $K_{I_0}(\theta) = (\pi + \alpha t_0)\sin\theta$, when $\arctan\frac{t_0}{\pi+\alpha t_0} \le \theta < \frac{\pi}{2} - \beta$, hence

$$\liminf_{\theta\to\frac{\pi}{2}-\beta} \frac{\pi\cos\beta - K_{I_0}(\theta)}{\frac{\pi}{2} - |\theta+\beta|} = (\pi + \alpha t_0)\sin\beta + t_0\cos\beta = \pi\sin\beta + \frac{t_0}{\cos\beta} < \pi\sin\beta + \frac{\log\rho}{\cos\beta},$$

which is a contradiction.

So we have proved that (38) holds. From this it follows that there exists a function $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ such that

a) $\varepsilon \in C(\mathbb{R}^+)$,

b) $\varepsilon(t) > 0, \quad t \in [0, \log\rho)$ and $\varepsilon(t) = 0, \quad t \in [\log\rho, \infty)$,

c) ε is decreasing,

d) $\{-t + is : \alpha t - \pi + \varepsilon(t) \le s \le \alpha t + \pi - \varepsilon(t),\ t \ge 0\} \supset I_\varphi$.

Consider a function γ given by the formula

$$\gamma(t) = \frac{2}{\log^2\rho}\int_t^\infty (\tau - t)\varepsilon(\tau)d\tau, \quad t \in \mathbb{R}^+. \tag{40}$$

Lemma 3. *The function γ has the following properties*

a) $\gamma \in C^2(\mathbb{R}^+)$,

b) $0 < \gamma(t) \le \varepsilon(t), \quad t \in [0, \log\rho)$ *and* $\gamma(t) = 0, \quad t \in [\log\rho, \infty)$, *c)* γ *is convex,*

d) $\{-t+is : \alpha t-\pi+\gamma(t)\leq s\leq \alpha t+\pi-\gamma(t),\ t\geq 0\}\supset I_\varphi$.

Proof of the Lemma 3. The statements of the items $a)$ and $c)$ follow from the equality $\gamma''(t)=\frac{2\varepsilon(t)}{\log^2\rho}$ and from the properties of the function ε.

Now let us prove that $\gamma(t)\leq\varepsilon(t)$ for all $t\in[0,\log\rho)$. Indeed, for any such a t we have

$$\begin{aligned}\gamma(t)&=\frac{2}{\log^2\rho}\int_t^\infty(\tau-t)\varepsilon(\tau)d\tau=\frac{2}{\log^2\rho}\int_t^{\log\rho}(\tau-t)\varepsilon(\tau)d\tau\\&\leq\frac{2\varepsilon(t)}{\log^2\rho}\int_t^{\log\rho}(\tau-t)d\tau=\frac{2\varepsilon(t)}{\log^2\rho}\frac{(\log\rho-t)^2}{2}\leq\varepsilon(t).\end{aligned}$$

The inequality $\gamma(t)\leq\varepsilon(t)$ yields that

$$\begin{aligned}&\{-t+is : \alpha t-\pi+\gamma(t)\leq s\leq\alpha t+\pi-\gamma(t),\ t\geq 0\}\supset\\&\quad\{-t+is : \alpha t-\pi+\varepsilon(t)\leq s\leq\alpha t+\pi-\varepsilon(t),\ t\geq 0\},\end{aligned}$$

hence by the property $d)$ of the function ε we have that

$$\{-t+is : \alpha t-\pi+\gamma(t)\leq s\leq\alpha t+\pi-\gamma(t), t\geq 0\}\supset I_\varphi.$$

The lemma is proven. □

Consider the sets

$$L=\{t+is\ |\alpha t-\pi+\gamma(t)\leq s\leq\alpha t+\pi-\gamma(t),\ t\in[0,\infty)\},$$

$$E=\exp(L)=\overline{\mathbb{C}}\setminus(D_1(0)\cup\{te^{i(\alpha\log t+\pi+\theta)}\ |\ |\theta|<\gamma(\log t), t\in[1,\rho)\}).$$

From the convexity of the function γ it follows that the set L is convex, thus E is log-convex. Note that $L=L(E)$ and the direction of the set L is $\beta=\arctan\alpha$. Furthermore, since we have that

$$-L^*(E)=\{-t+is\ |\alpha t-\pi+\gamma(t)\leq s\leq\alpha t+\pi-\gamma(t),\ t\geq 0\},$$

by the property $d)$ of the function γ, we obtain that

$$I_\varphi\subset -L^*(E).$$

Now applying Theorem 2 we get that the series (2) defines an analytic function $f\in H(\overline{\mathbb{C}}\setminus E)$. It remains to see that since $\gamma(t)>0$ on the interval $[0,\log\rho)$, we have that

$$\overline{\mathbb{C}}\setminus E=D_1(0)\cup\{te^{i(\alpha\log t+\pi+\theta)}\ |\ |\theta|<\gamma(\log t),\ t\in[1,\rho)\}\supset\gamma_\rho,$$

and therefore the element (2) has an analytic continuation along the segment γ_ρ of α-spiral passing through the point -1, hence Theorem 1 is fully proven. □

3 Examples of power series

As it was pointed out in the introduction, one of the issues of the theory of the analytic continuations is the localization of the singularities of the element (2). In current work we are interested in the localization of the singularities of a power series outside of its disc of convergence.

More precisely, for every $\alpha \in \mathbb{R}$, we would like to find the smallest $\rho \geq 1$ (in terms of the coefficients $\{f_n\}$) such that the series (2) do not continue analytically along the segment $\overline{\gamma_\rho}$ (see (9)). In other words, we would like to find the first singularity of the power series (2) along the α-spiral passing through the point -1.

The Mittag-Leffler α-star of an element (2) is the maximal α-starlike domain, where the element continues analytically. For any angle $\theta \in [0, 2\pi)$ denote by

$$\gamma_f(\theta) = \{e^{i\theta} t^{1+i\alpha} \mid t \in [1, \rho_f(\theta))\}, \tag{41}$$

where

$$\rho_f(\theta) = \sup\{\rho' \mid f \in H(\gamma_{\rho'})\}. \tag{42}$$

We call $\gamma_f(\theta)$ the θ direction *wing* of the Mittag-Leffler α-star of the element (2) and the number $\rho_f(\theta)$ the "*length*" of the wing $\gamma_f(\theta)$ (note that it is not the length of the curve $\gamma_f(\theta)$ and it is a slight abuse of a term, but it will not cause any confusion).

A simple notion to characterize singularities of a power series is the notion of *lacunas* of the series. This is the zero set of its coefficients. The density of the lacunas has been proven to be an effective way to localize the singularities on the boundary of the convergence disc. For a detailed discussion on these see [4].

It turns out that in the case of analytic continuation along the spirals outside of the disc of convergence lacunas do not play a role and we address this issue in this section.

For a set $P \subset \mathbb{N}_0$ we say that it has a density if there exists a limit

$$\Delta(P) = \lim_{r\to\infty} \frac{|\{n \in P \mid n \leq r\}|}{r}. \tag{43}$$

We call it density of P and denote by $\Delta(P)$.

For a series (1) we denote by P_f its lacunas:

$$P_f = \{n \mid f_n = 0\}. \tag{44}$$

Proposition 1. *Fix any number $\alpha \in \mathbb{R}$ and $\rho > 1$. Then for any given number $0 \leq \Delta < 1$ and any set $P_0 = \{p_n\}_{n=0}^{\infty} \subset \mathbb{N}_0$ that has density $\Delta(P_0) = \Delta$, there exists a power series (2) such that its Mittag-Leffler α-star contains wings of "length" ρ and its lacunas are the set P_0.*

To prove this proposition we need the following theorem

Theorem 3 ([4])**.** *The element* (2) *can be continued analytically to the whole complex plane except possibly the arc* $\{z \mid |z| = 1,\ |\arg z| \le \sigma\}$, *where* $\sigma \in [0, \pi)$, *if and only if there exists a function* $\varphi \in H(\mathbb{C})$ *of the finite exponential type for which*

$$\varphi(n) = f_n, \quad \forall n \in \mathbb{N}_0$$

and

$$h_\varphi(\theta) \le \sigma |\sin \theta|, \quad |\theta| \le \pi. \tag{45}$$

Remark 3. *This theorem is a simple application of the Theorem 2.*

Proof of the Proposition 1. Consider following function

$$\varphi_{P_0}(z) = \prod_{n \in P_0} (1 - \frac{z^2}{n^2}) \tag{46}$$

and the power series $g(z) = \sum_{n=0}^{\infty} g_n z^n$, where

$$g_n = \varphi_{P_0}(n). \tag{47}$$

Since the density of P_0 is Δ, we have that (see [5])

$$h_{\varphi_{P_0}}(\theta) = \pi\Delta |\sin \theta|, \quad |\theta| \le \pi. \tag{48}$$

From equations (47) and (48) it follows that the radius of convergence of g is 1 and it satisfies the hypothesis of the Theorem 3 with $\sigma = \pi\Delta$. Hence g can be analytically continued to $\mathbb{C} \setminus \{z \mid |z| = 1,\ |\arg z| \le \pi\Delta\}$.

Furthermore, the lacunas of g are $P_g = P_0$. Since $\Delta(\mathbb{N}_0 \setminus P_0) = 1 - \Delta(P_0)$, we can choose a infinite subset $Q \subset \mathbb{N}_0 \setminus P_0$ such that

$$\Delta(Q) = 0. \tag{49}$$

Now consider the series $q(z) = \sum\limits_{n=0}^{\infty} q_n z^n$, where

$$\begin{cases} q_n = \rho^{-n},\ n \in Q, \\ q_n = 0,\ n \overline{\in} Q \end{cases} \tag{50}$$

We have that

$$\limsup_{n \to \infty} |q_n|^{\frac{1}{n}} = \lim_{n \in Q} |q_n|^{\frac{1}{n}} = \rho^{-1},$$

hence the convergence disc of q is $D_\rho(0)$. Furthermore, $P_q = \mathbb{N}_0 \setminus Q$, thus from (49) it follows that

$$\Delta(P_q) = \Delta(\mathbb{N}_0 \setminus Q) = 1 - \Delta(Q) = 1,$$

which means that q has lacunas of full density 1, therefore it cannot be continued analytically to any point outside of the disc $D_\rho(0)$ (see [4]). Let h be the function

$$h(z) = g(z) + q(z) \tag{51}$$

and

$$h(z) = \sum_{n=0}^{\infty} h_n z^n, \tag{52}$$

be its expansion.

Lemma 4. *a)* $h \in H(D_\rho(0) \setminus \{z \mid |z| = 1,\ |\arg z| \leq \pi\Delta\})$,

b) the raduis of convergence of the element (52) *is* 1,

c) $P_h \supset P_0$.

Proof of the Lemma 4. The item a) is a consequence of the facts that

$$g \in H(\overline{\mathbb{C}} \setminus \{z \mid |z| = 1,\ |\arg z| \leq \pi\Delta\}) \tag{53}$$

and $q \in H(D_\rho(0))$.

To prove b) note that since $h \in H(D_\rho(0) \setminus \{z \mid |z| = 1,\ |\arg z| \leq \pi\Delta\})$, then $h \in H(D_1(0))$. On the other hand since the disc of convergence of g is $D_1(0)$, it has to have at least one singularity on the boundary $\partial D_1(0)$, but q is analytic on $\overline{D_1(0)}$, hence that singularity is also a singularity for h. This means that the disc of convergence of h is $D_1(0)$ or equivalently the radius of convergence of the series (52) is 1.

It remains to prove the item c). We have that $P_g = P_0$ and $P_q = \mathbb{N}_0 \setminus Q \supset P_0$, therefore $g_n = q_n = 0,\ n \in P_0$ and consequently $h_n = 0,\ n \in P_0$ or $P_h \supset P_0$.

Lemma is fully proven. □

Let us introduce another power series

$$r(z) = \sum_{n=0}^{\infty} r_n z^n, \tag{54}$$

where

$$\begin{cases} r_n = \frac{1}{n^n}, n \in P_h \setminus P_0 \\ r_n = 0, n \in \mathbb{N}_0 \setminus (P_h \setminus P_0). \end{cases} \tag{55}$$

The function r is an entire function because $\lim\limits_{n\to\infty} |r_n|^{1/n} = 0$.

Finally, consider the function

$$f(z) = h(z) + r(z), \tag{56}$$

with expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n. \tag{57}$$

Lemma 5. *a)* $f \in H(D_\rho(0) \setminus \{z \mid |z| = 1,\ |\arg z| \leq \pi\Delta\})$,

b) the radius of convergence of the series (57) *is* 1,

c) $P_f = P_0$*:*

Proof of the Lemma 5. The assertions of the items *a*), *b*) follow from the ones of the items *a*), *b*) of the Lemma 4 and the fact that r is an entire function.

Since $r_n = 0$, when $n \in P_0$ or $n \in \mathbb{N}_0 \setminus P_h$, we have that $f_n = h_n = 0$ for $n \in P_0$ and $f_n = h_n \neq 0$ for $n \in \mathbb{N}_0 \setminus P_h$. And finally, for $n \in P_h \setminus P_0$ we have that $h_n = 0$ and $r_n = 1/n^n$, hence $f_n = 1/n^n \neq 0$. This proves the assertion of item *c*) and the lemma is fully proven. □

To finish the proof of the Proposition 1, it remains to see that the element (57) is the one we need. Indeed, from the previous lemma we know that it has a radius of convergence 1, lacunas P_0 and is analytic on the domain $D_\rho(0) \setminus \{z \mid |z| = 1,\ |\arg z| \leq \pi\Delta\}$. Therefore, we can say that

$$\rho_f(\theta) \leq \rho, \quad |\theta| \leq \pi\Delta \tag{58}$$

and

$$\rho_f(\theta) = \rho, \quad |\theta| > \pi\Delta. \tag{59}$$

□

4 Natural zeros of the interpolating entire function

In this section we discuss the balance in between the growth of the entire function $\varphi \in B(\Pi_{-\beta})$ that satisfies the conditions (11), (12) and the density of its natural zeros, i.e. zeros that are natural numbers.

Theorem 4. *Let* $\varphi \in B(\Pi_{-\beta})$ *is such that* $h_\varphi(0) = 0$ *and bounds* (11), (12) *hold. Then* $\{n \in \mathbb{N}_0 \mid \varphi(n) = 0\} \neq \mathbb{N}_0$.

Proof. Let us assume that on contrary:

$$\varphi(n) = 0, \quad \forall n \in \mathbb{N}_0. \tag{60}$$

Consider the function $\varphi_0(z) = \sin \pi z$. The function φ_0 is a function of a finite exponential type, such that all integer numbers are its simple zeros:

$$\varphi_0(z) = \pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}). \tag{61}$$

We have that

$$h_{\varphi_0}(\theta) = \pi |\sin \theta|, \quad |\theta| \leq \pi, \tag{62}$$

hence the function

$$\psi(z) = \frac{\varphi(z)}{\varphi_0(z)}, \quad z \in \Pi_{-\beta} \tag{63}$$

is of a finite exponential type and moreover,

$$h_\psi(\theta) = h_\varphi(\theta) - \pi|\sin\theta|, \quad |\theta + \beta| < \frac{\pi}{2}. \tag{64}$$

Since h_ψ is trigonometrically convex, we have that

$$h_\psi(0)\sin(\pi - 2\theta) \le h_\psi(-\frac{\pi}{2} - \beta + \theta)\sin(\frac{\pi}{2} - \beta - \theta) + h_\psi(\frac{\pi}{2} - \beta - \theta)\sin(\frac{\pi}{2} + \beta - \theta), \tag{65}$$

for $0 < \theta < \min(\pi/2 - \beta, \pi/2 + \beta)$.

Pick some $1 < \rho' < \rho$, then from the inequalities (11), (12) we have that

$$h_\varphi(\frac{\pi}{2} - \beta - \theta) \le \pi\cos\beta - (\pi\sin\beta + \frac{\log\rho'}{\cos\beta})\theta \tag{66}$$

and

$$h_\varphi(-\frac{\pi}{2} - \beta + \theta) \le \pi\cos\beta - (-\pi\sin\beta + \frac{\log\rho'}{\cos\beta})\theta. \tag{67}$$

From these we obtain

$$h_\psi(\frac{\pi}{2} - \beta - \theta) \le \pi\cos\beta - (\pi\sin\beta + \frac{\log\rho'}{\cos\beta})\theta - \pi\cos(\beta + \theta) \tag{68}$$

and

$$h_\psi(-\frac{\pi}{2} - \beta + \theta) \le \pi\cos\beta - (-\pi\sin\beta + \frac{\log\rho'}{\cos\beta})\theta - \pi\cos(\beta - \theta). \tag{69}$$

Thus, from (65), we get that

$$\begin{aligned}\frac{h_\psi(0)\sin 2\theta}{\theta} \le &(\frac{\pi\cos\beta - \pi\cos(\beta+\theta)}{\theta} - \pi\sin\beta - \frac{\log\rho'}{\cos\beta})\cos(\beta - \theta)\\ &+ (\frac{\pi\cos\beta - \pi\cos(\beta-\theta)}{\theta} + \pi\sin\beta - \frac{\log\rho'}{\cos\beta})\cos(\beta + \theta),\end{aligned}$$

for sufficiently small θ-s. Now passing to the limit when θ goes to zero, we obtain that

$$h_\psi(0) \le -\log\rho' < 0, \tag{70}$$

which is a contradiction, because $h_\psi(0) = h_\varphi(0) = 0$. □

Theorem 5. *Let $P_0 \subset \mathbb{N}_0$ be a subset with a density $0 \le \Delta = \Delta(P_0) < 1$. Then, for every $\rho > 1$ and $\beta \in (-\pi/2, \pi/2)$ there exists a function $\varphi \in B(\Pi_{-\beta})$ such that*

$$\{n \in \mathbb{N}_0 : \varphi(n) = 0\} = P_0, \tag{71}$$

$$\lim_{\theta \to \frac{\pi}{2} - \beta} \frac{\pi\cos\beta - h_\varphi(\theta)}{\frac{\pi}{2} - |\theta + \beta|} = \pi\sin\beta + \frac{\log\rho}{\cos\beta}, \tag{72}$$

$$\lim_{\theta \to -\frac{\pi}{2} - \beta} \frac{\pi\cos\beta - h_\varphi(\theta)}{\frac{\pi}{2} - |\theta + \beta|} = -\pi\sin\beta + \frac{\log\rho}{\cos\beta}. \tag{73}$$

Proof. Consider the function

$$\varphi(z) = \varphi_{P_0}(z) \sin\left(\sigma_1 z e^{i(\beta-\theta_0)}\right) \sin\left(\sigma_2 z e^{i(\beta+\theta_0)}\right) e^{-(\sigma_3+i\sigma_4)z}, \quad z \in \mathbb{C}, \tag{74}$$

where

$$\begin{cases} \theta_0 = \arctan \frac{\log\rho+\sqrt{\log^2\rho+(\pi(1-\Delta)\sin 2\beta)^2}}{2\pi(1-\Delta)\cos^2\beta}, \\ \sigma_1 = \frac{\pi(1-\Delta)\sin(\theta_0+\beta)}{\sin 2\theta_0}, \\ \sigma_2 = \frac{\pi(1-\Delta)\sin(\theta_0-\beta)}{\sin 2\theta_0}, \\ \sigma_3 = \log\rho, \\ \sigma_4 = \log\rho\tan\beta, \end{cases} \tag{75}$$

and

$$\varphi_{P_0} = \prod_{n\in P_0} \left(1 - z^2/n^2\right). \tag{76}$$

Note that the zeros of φ and φ_0 are the same, hence (71) holds.

Furthermore,

$$\tan\theta_0 = \frac{\log\rho + \sqrt{\log^2\rho + (\pi(1-\Delta)\sin 2\beta)^2}}{2\pi(1-\Delta)\cos^2\beta} > \frac{\sqrt{(\pi(1-\Delta)\sin 2\beta)^2}}{2\pi(1-\Delta)\cos^2\beta} = \tan|\beta|,$$

thus $\theta_0 > |\beta|$, so $\sigma_1, \sigma_2 > 0$ and therefore from the equation (48) we have that

$$h_\varphi(\theta) = \pi\Delta|\sin\theta| + \sigma_1|\sin(\theta+\beta-\theta_0)| + \sigma_2|\sin(\theta+\beta+\theta_0)| - \sigma_3\cos\theta + \sigma_4\sin\theta.$$

A straightforward computation shows that $h_\varphi(\theta)$ satisfies the conditions (72) and (73). □

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