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Global well-posedness for the 3D Newton-Boussinesq equations

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Abstract. In this paper, we prove the global well-posedness for the Newton-Boussinesq equations in $\mathbb{R}^3$. By fully using the Fourier localization technique, we obtain the existence and uniqueness of smooth solutions in Hilbert spaces.

Key Words: Global well-posedness; Newton-Boussinesq equations.
Mathematics Subject Classification 2010: 35Q35

1 Introduction

In this paper we consider the following 3D Newton-Boussinesq equations:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \nabla \times (\theta e_3), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t \theta + u \cdot \nabla \theta - \Delta \theta = 0, \\ \nabla \cdot u = 0, \\ (u, \theta)|_{t=0} = (u_0, \theta_0). \end{cases} \tag{1}$$

Here, the vector $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ stands for the fluid velocity field, the scalar function θ denotes temperature, $\omega = \nabla \times u$ is the vorticity of the fluid, while ω_0 and θ_0 are the initial data. According to [14], we find that $\omega = \nabla \times u$ is the vorticity of the fluid and the explicit formula for u is

$$u(x) = \int_{\mathbb{R}^3} K(x-y)\omega(y)dy,$$

where the 3×3 matrix kernel K is

$$K(x)h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}, \quad h \in \mathbb{R}^3.$$

It is well known that Newton-Boussinesq system is widely used to model the dynamics of the ocean or the atmosphere. It arises from the density dependent fluid equations by using the so-called Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity. This approximation can be justified from compressible fluid equations by a simultaneous low Mach number/Froude number limit, we refer to [6] for a rigorous justification.

Comparing the system (1) with the 3D Boussinesq equation in vorticity form,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u + \nabla \times (\theta e_3), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t \theta + u \cdot \nabla \theta - \Delta \theta = 0, \\ \nabla \cdot u = 0, \\ (u,\theta)|_{t=0} = (u_0, \theta_0), \end{cases} \tag{2}$$

we find that the vortex-stretching term $\omega \cdot \nabla u$ vanishes in the first equation of (1). It is well-known that the term $\omega \cdot \nabla u$ is difficult to control in three dimensional case. Because we do not know how the term affects the dynamic of the fluid, the global well-posedness for the 3D Boussinesq equation is still an open problem.

Once the term $\omega \cdot \nabla u$ is removed, the 3D Boussinesq equation becomes easier to study. The most important model is 3D Newton-Boussinesq equation (1) which arises from the study of Bénard flow [3]. As for (1), some developments have been achieved by many authors. For the 2D case, using the spectral methods and the nonlinear Galerkin methods, the existence and uniqueness of the solutions for (1) were obtained by Guo [9, 10]. Then the asymtotic behavior and a blow-up criterion of (1) was investigated in [7] and [19] respectively. As for 3D case, some regularity criteria for (1) were established in [8, 21].

Since the 3D Newton-Boussinesq system is similar to the 2D Boussinesq equations, we also recall some noticeable works about the 2D Boussinesq system. Chae [2], and Hou and Li [12] established the global well-posedness of the smooth solution for the system. Then the result was extended by Hmidi and Keraani [11] with rough initial data, namely $u_0 \in B^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ and $\theta_0 \in L^p(\mathbb{R}^2)$ for $p > 2$. Danchin and Paicu [5] proved the global well-posedness for the 2D Boussinesq system with less regular initial data satisfying $(u_0, \theta_0) \in L^2(\mathbb{R}^2)$ and $\omega_0 = \nabla \times u_0 \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. There are a lot of papers devoted to studying Boussinesq equations, interested readers can refer to [15, 16, 17].

In this paper, we aim at to establish the global well-posedness for the 3D Newton-Boussinesq equations in Hilbert spaces. Using the Fourier localization technique, we obtain the existence and uniqueness of smooth solutions for (1). Here we follow ideas introduced in [4]. Our main result is as following.

Theorem 1.1 *Let* $(\omega_0, \theta_0) \in (H^s \times H^s)$ *with* $s > 3$. *Then* $\forall\ T > 0$, *the initial value problem* (1) *has a unique solution* (ω, θ) *satisfying*

$$\omega \in \mathcal{C}(\mathbb{R}^+; H^s) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^{s+1}) \quad \textit{and} \quad \theta \in \mathcal{C}(\mathbb{R}^+; H^s) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^{s+1}). \tag{3}$$

Notation: Throughout the paper, C stands for a generic constant, and changes from line to line; C_T means a constant C depending on T; $\|\cdot\|_p$ denotes the norm of the Lebesgue space L^p.

2 Preliminaries

In this preparatory section, we provide the definition of some function spaces and review some important lemmas.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^d)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

We now denote the operator $(I - \Delta)^{\frac{1}{2}}$ by Λ which is defined by

$$\widehat{\Lambda f}(\xi) = (1 + |\xi|^2)^{\frac{1}{2}} \hat{f}(\xi).$$

More generally, $\Lambda^s f$ for $s \in \mathbb{R}$ can be identified with the Fourier Transform

$$\widehat{\Lambda^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).$$

For $s \in \mathbb{R}$, we define

$$\|f\|_{H^s} \triangleq \|\Lambda^s f\|_{L^2} \triangleq \left(\int_{\mathbb{R}^n} (1 + |\xi|^s)^2 |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

and the Sobolev space H^s is denoted by $H^s \triangleq \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{H^s} < \infty\}$. The usual Sobolev space $H^{s,p}$ is endowed with the norm

$$\|f\|_{H^{s,p}} \triangleq \|\Lambda^s f\|_{L^p}.$$

We can refer to [20] for more details.

Lemma 2.1 *[18] Suppose that the vector function* f *is divergence-free and set* $g = \nabla \times f$. *Then there exists a constant* C *independent of* f *such that*

$$\|\nabla f\|_p \le C\|g\|_p, \quad \forall\ 1 < p < \infty. \tag{1}$$

Lemma 2.2 *[13] Let* $1 < p < \infty$, $s > 0$. *Assume that* $f, g \in H^{s,p}$, *then there exist constants* C *independent of* f, g *such that*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}} \|g\|_{L^{p_4}}) \tag{2}$$

with $p_2, p_3 \in (1, +\infty)$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

3 Proof of the main result

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 Step 1. A priori estimate.

L^2 **estimates.**

Taking the L^2-inner product of the first equation of (1) with ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2+\|\nabla\omega\|_2^2 &= \int_{\mathbb{R}^3}\nabla\times(\theta e_3)\omega dx\\ &\le \|\theta\|_2\|\nabla\omega\|_2\\ &\le C\|\theta\|_2^2+\frac{1}{2}\|\nabla\omega\|_2^2, \qquad (1)\end{aligned}$$

from which we obtain

$$\frac{d}{dt}\|\omega\|_2^2+\|\nabla\omega\|_2^2\le C\|\theta\|_2^2. \qquad (2)$$

Similarly we get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2+\|\nabla\theta\|_2^2\le 0. \qquad (3)$$

Summing up (2)-(3) and using the Gronwall inequality, we have

$$\|\omega(t)\|_2^2+\|\theta(t)\|_2^2+\int_0^t(\|\nabla\omega(\tau)\|_2^2+\|\nabla\theta(\tau)\|_2^2)d\tau\le C(\|\omega_0\|_2^2+\|\theta_0\|_2^2)e^{Ct}. \quad (4)$$

H^1 **estimates.**

Applying ∇ to the first equation of (1), multiplying $\nabla\omega$ with the obtaining equality and integrating with respect to x variable, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_2^2+\|\Delta\omega\|_2^2 &= -\int_{\mathbb{R}^3}\nabla u\cdot\nabla\omega\nabla\omega dx+\int_{\mathbb{R}^3}\nabla(\nabla\times(\theta e_3))\nabla\omega dx\\ &\le \|\nabla u\|_2\|\nabla\omega\|_4^2+\|\nabla\theta\|_2\|\Delta\omega\|_2\\ &\le \|\omega\|_2\|\nabla\omega\|_2^{\frac{1}{2}}\|\Delta\omega\|_2^{\frac{3}{2}}+C\|\nabla\theta\|_2^2+\frac{1}{4}\|\Delta\omega\|_2^2\\ &\le \|\omega\|_2^4\|\nabla\omega\|_2^2+\frac{1}{4}\|\Delta\omega\|_2^2+C\|\nabla\theta\|_2^2+\frac{1}{4}\|\Delta\omega\|_2^2, (5)\end{aligned}$$

from which we get

$$\frac{d}{dt}\|\nabla\omega\|_2^2+\|\Delta\omega\|_2^2\le\|\omega\|_2^4\|\nabla\omega\|_2^2+C\|\nabla\theta\|_2^2. \qquad (6)$$

Similarly we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2+\|\Delta\theta\|_2^2 &= -\int_{\mathbb{R}^3}\nabla u\cdot\nabla\theta\nabla\theta dx\\ &\le \|\nabla u\|_2\|\nabla\theta\|_4^2\\ &\le \|\omega\|_2\|\nabla\theta\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{3}{2}}\\ &\le C\|\omega\|_2^4\|\nabla\theta\|_2^2+\frac{1}{2}\|\Delta\theta\|_2^2, \end{aligned}\tag{7}$$

from which we obtain

$$\frac{d}{dt}\|\nabla\theta\|_2^2+\|\Delta\theta\|_2^2\le C\|\omega\|_2^4\|\nabla\theta\|_2^2. \tag{8}$$

Summing up (6) and (8) and using the Gronwall inequality yield

$$\begin{aligned}&\|\nabla\omega(t)\|_2^2+\|\nabla\theta\|_2^2+\int_0^t(\|\Delta\omega(\tau)\|_2^2+\|\Delta\theta(\tau)\|_2^2)d\tau\\ &\le C(\|\omega_0\|_{H^1}+\|\theta_0\|_{H^1})e^{Ct}. \end{aligned}\tag{9}$$

***H^s* estimates.**

Applying Λ^s to the first equation of (1), we have

$$\partial_t\Lambda^s\omega+u\cdot\nabla\Lambda^s\omega-\Delta\Lambda^s\omega=u\cdot\nabla\Lambda^s\omega-\Lambda^s(u\cdot\nabla\omega)-\nabla\times\Lambda^s(\theta e_3). \tag{10}$$

Multiplying the above equality by $\Lambda^s\omega$ and using Lemma 2.1 and 2.2 yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\Lambda^s\omega\|_2^2+\|\Lambda^{s+1}\omega\|_2^2\\ \le\ & \|u\cdot\nabla\Lambda^s\omega-\Lambda^s(u\cdot\nabla\omega)\|_2\|\Lambda^s\omega\|_2+\|\Lambda^s\theta\|_2\|\Lambda^{s+1}\omega\|_2\\ \le\ & (\|\nabla u\|_4\|\nabla\omega\|_{H^{s-1,4}}+\|u\|_{H^{s,6}}\|\nabla\omega\|_3)\|\Lambda^s\omega\|_2\\ & +C\|\Lambda^s\theta\|_2^2+\frac{1}{4}\|\Lambda^{s+1}\omega\|_2^2\\ \le\ & \|\omega\|_2^{\frac{1}{4}}\|\nabla\omega\|_2^{\frac{3}{4}}\|\omega\|_{H^s}^{\frac{5}{4}}\|\omega\|_{H^{s+1}}^{\frac{3}{4}}\\ & +\|\omega\|_{H^{s+1}}\|\nabla\omega\|_2^{\frac{1}{2}}\|\Delta\omega\|_2^{\frac{1}{2}}\|\Lambda^s\omega\|_2+C\|\Lambda^s\theta\|_2^2+\frac{1}{4}\|\Lambda^{s+1}\omega\|_2^2\\ \le\ & C\|\omega\|_2^{\frac{2}{5}}\|\nabla\omega\|_2^{\frac{6}{5}}\|\omega\|_{H^s}^2+\frac{1}{4}\|\omega\|_{H^{s+1}}^2+\|\omega\|_{H^s}^2\|\nabla\omega\|_2\|\Delta\omega\|_2\\ & +C\|\Lambda^s\theta\|_2^2+\frac{1}{4}\|\Lambda^{s+1}\omega\|_2^2, \end{aligned}\tag{11}$$

from which we get

$$\begin{aligned}&\frac{d}{dt}\|\omega\|_{H^s}^2+\|\omega\|_{H^{s+1}}^2\\ \le\ & C(\|\omega\|_2^{\frac{2}{5}}\|\nabla\omega\|_2^{\frac{6}{5}}+\|\nabla\omega\|_2^{\frac{1}{4}}\|\Delta\omega\|_2^{\frac{3}{4}})\|\omega\|_{H^s}^2+C\|\theta\|_{H^s}^2. \end{aligned}\tag{12}$$

Similarly we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^s\theta\|_2^2+\|\Lambda^{s+1}\theta\|_2^2\\
\leq\ & \|u\cdot\nabla\Lambda^s\theta-\Lambda^s(u\cdot\nabla\theta)\|_2\|\Lambda^s\theta\|_2\\
\leq\ & (\|\nabla u\|_4\|\nabla\theta\|_{H^{s-1,4}}+\|u\|_{H^{s,4}}\|\nabla\theta\|_4)\|\Lambda^s\theta\|_2\\
\leq\ & \|\omega\|_2^{\frac{1}{4}}\|\nabla\omega\|_2^{\frac{3}{4}}\|\theta\|_{H^s}^{\frac{5}{4}}\|\theta\|_{H^{s+1}}^{\frac{3}{4}}+\|u\|_{H^s}^{\frac{1}{4}}\|\nabla u\|_{H^s}^{\frac{3}{4}}\|\nabla\theta\|_2^{\frac{1}{4}}\|\Delta\theta\|_2^{\frac{3}{4}}\|\Lambda^s\theta\|_2\\
\leq\ & C\|\omega\|_2^{\frac{2}{5}}\|\nabla\omega\|_2^{\frac{6}{5}}\|\theta\|_{H^s}^2+\frac{1}{4}\|\theta\|_{H^{s+1}}^2\\
&+\|\omega\|_{H^s}^2\|\nabla\theta\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{3}{2}}+C\|\Lambda^s\theta\|_2^2, \qquad (13)
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\theta\|_{H^s}^2+\|\theta\|_{H^{s+1}}^2\\
\leq\ & \|\omega\|_2^{\frac{2}{5}}\|\nabla\omega\|_2^{\frac{6}{5}}\|\theta\|_{H^s}^2+\|\omega\|_{H^s}^2\|\nabla\theta\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{3}{2}}+C\|\theta\|_{H^s}^2. \qquad (14)
\end{aligned}$$

Summing up (12) and (14) and using the Gronwall inequality, we have

$$\begin{aligned}
&\|\omega(t)\|_{H^s}+\|\theta(t)\|_{H^s}+\int_0^t(\|\omega(\tau)\|_{H^{s+1}}^2+\|\theta(\tau)\|_{H^{s+1}}^2)d\tau\\
\leq\ & C(\|\omega_0\|_{H^s}+\|\theta_0\|_{H^s})e^{e^{Ct}}. \qquad (15)
\end{aligned}$$

Step 2. Existence.

We smooth the data to get the following approximate system

$$\begin{cases}
\partial_t\omega^n+u^n\cdot\nabla\omega^n=\nabla\times(\theta^n e_3), \quad n\in\mathbb{N},\\
\partial_t\theta^n+u^n\cdot\nabla\theta^n-\Delta\theta^n=0,\\
\nabla\cdot u^n=0,\\
(\omega,\theta)|_{t=0}=(\mathcal{J}_n\omega_0,\mathcal{J}_n\theta_0),
\end{cases} \qquad (16)$$

where the cutoff operator $\mathcal{J}_n$ is defined as

$$(\mathcal{J}_nf)(x)=\mathcal{F}^{-1}(\hat{f}(\cdot)1_{B_n}(\cdot))(x)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}\hat{f}(\xi)1_{\{|\cdot|\leq n\}}(\xi)d\xi.$$

The classical theory of quasi-linear hyperbolic systems (cf. [1]) guarantees that the approximate system has a unique smooth solution (ω^n,θ^n) on $[0,T]$ if the initial data $\mathcal{J}_n\omega_0$, $\mathcal{J}_n\theta_0\in H^s$ for every $s\in\mathbb{R}$. Moreover a priori estimates (15) ensures that the solution (ω^n,θ^n) is global. Since $\|\mathcal{J}_n\omega_0\|_{H^s}\leq\|\omega_0\|_{H^s}$, $\|\mathcal{J}_n\theta_0\|_{H^s}\leq\|\theta_0\|_{H^s}$, then a priori estimates (15) for system (16) are uniform in n, that is,

$$\omega^n\in L^\infty_{\rm loc}(\mathbb{R}^+,H^s)\cap L^2_{\rm loc}(\mathbb{R}^+;H^{s+1}) \quad\text{and}\quad \theta^n\in L^\infty_{\rm loc}(\mathbb{R}^+;H^s)\cap L^2_{\rm loc}(\mathbb{R}^+;H^s).$$

In order to show the convergence, we also need uniform boundedness for $\partial_t\omega^n$ and $\partial_t\theta^n$. From the first equation of (16), we know

$$\begin{aligned}\|\partial_t\omega^n\|_{L_t^\infty H^{-1}} &\le \|u^n\cdot\nabla\omega^n\|_{L_t^\infty H^{-1}}+\|\nabla\theta^n\|_{L_t^\infty H^{-1}}\\ &\le \|u^n\omega^n\|_{L_t^\infty L^2}+\|\theta^n\|_{L_t^\infty L^2}\\ &\le \|u^n\|_{L_t^\infty H^s}\|\omega^n\|_{L_t^\infty H^s}+\|\theta^n\|_{L_t^\infty H^s}\\ &\le C. \end{aligned}\tag{17}$$

Similarly we have

$$\begin{aligned}\|\partial_t\theta^n\|_{L_t^\infty H^{-1}} &\le \|u^n\cdot\nabla\theta^n\|_{L_t^\infty H^{-1}}+\|\Delta\theta^n\|_{L_t^\infty H^{-1}}\\ &\le \|u\theta\|_{L_t^\infty L^2}+\|\theta\|_{L_t^\infty H^1}\\ &\le \|u\|_{L_t^\infty H^s}\|\theta\|_{L_t^\infty H^s}+\|\theta\|_{L_t^\infty H^s}\\ &\le C. \end{aligned}\tag{18}$$

Since H^{-1} is locally compactly embedded in L^2, the classical Aubin-Lions argument ensures that, up to an extraction of subsequence, the approximate solution sequence $(\omega^n,\theta^n)_{n\in\mathbb{N}}$ strongly converges in $L^\infty_{\text{loc}}(\mathbb{R}^+;H^{-1})$ to some function (ω,θ) such that

$$\omega\in L^\infty_{\text{loc}}(\mathbb{R}^+,H^s)\cap L^2_{\text{loc}}(\mathbb{R}^+;H^{s+1})\quad\text{and}\quad \theta\in L^\infty_{\text{loc}}(\mathbb{R}^+;H^s)\cap L^2_{\text{loc}}(\mathbb{R}^+;H^s).$$

Using the above estimates, it is easy to pass the limit in the approximate system (16) and (ω,θ) solves (1) in the sense of distribution. By a classical deduction [4], we get $\omega,\ \theta\in\mathcal{C}(\mathbb{R}^+;H^s)$.

Step 3. Uniqueness.

Let (ω_1,θ_1) and (ω_2,θ_2) be two solutions of the equations (1) with the same initial data and satisfy (3). Denote $\delta\omega=\omega_1-\omega_2$ and $\delta\theta=\theta_1-\theta_2$. Then we have the difference equations as follows

$$\begin{cases}\partial_t\delta\omega+\delta u\cdot\nabla\omega_1+u_2\cdot\nabla\delta\omega=\nabla\times(\delta\theta e_3),\\ \partial_t\delta\theta+\delta u\cdot\nabla\theta_1+u_2\cdot\nabla\delta\theta-\Delta\delta\theta=0.\end{cases}\tag{19}$$

Taking the L^2-inner product with the first equation of (19) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\delta\omega\|_2^2+\|\nabla\delta\omega\|_2^2\\ =&\ -\int_{\mathbb{R}^3}\delta u\cdot\nabla\omega_1\delta\omega dx+\int_{\mathbb{R}^3}\nabla\times(\delta\theta e_3)\delta\omega dx\\ \le&\ \|\delta u\|_6\|\nabla\omega_1\|_3\|\delta\omega\|_2+\|\delta\theta\|_2\|\nabla\delta\omega\|_2\\ \le&\ \|\delta\omega\|_2^2\|\nabla\omega_1\|_2^{\frac12}\|\Delta\omega_1\|_2^{\frac12}+\frac{1}{2}\|\nabla\delta\omega\|_2^2+C\|\delta\theta\|_2^2.\end{aligned}\tag{20}$$

It follows that

$$\frac{d}{dt}\|\delta\omega\|_2^2+\|\nabla\delta\omega\|_2^2\le\|\delta\omega\|_2^2\|\nabla\omega_1\|_2^{\frac12}\|\Delta\omega_1\|_2^{\frac12}+C\|\delta\theta\|_2^2.\tag{21}$$

Similarly we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\delta\theta\|_2^2+\|\nabla\delta\theta\|_2^2 &= -\int_{\mathbb{R}^3}\delta u\cdot\nabla\theta_1\delta\theta dx\\ &\leq \|\delta u\|_6\|\nabla\theta_1\|_3\|\delta\theta\|_2\\ &\leq \|\delta\omega\|_2^2\|\nabla\theta_1\|_2\|\Delta\theta\|_2+C\|\delta\theta\|_2^2. \end{aligned}\tag{22}$$

Summing up (21)-(22) and using the Gronwall inequality imply that $\omega_1 = \omega_2,\ \theta_1=\theta_2$.

Step 4. Continuous dependence with respect to the initial data.

Let $(\bar{\omega}_1,\bar{\theta}_1)$ and $(\bar{\omega}_2,\bar{\theta}_2)$ be two solutions of the equations (1) with the initial data $(\bar{\omega}_1^0,\bar{\theta}_1^0)$ and $(\bar{\omega}_2^0,\bar{\theta}_2^0)$ respectively such that

$$\|\bar{\omega}_1^0-\bar{\omega}_2^0\|_{H^s}\leq\epsilon \quad\text{and}\quad \|\bar{\theta}_1^0-\bar{\theta}_2^0\|_{H^s}\leq\epsilon,$$

where ϵ is a small constant. Similarly to the process of proving uniqueness, we have

$$\begin{aligned}\|\bar{\omega}_1-\bar{\omega}_2\|_{H^s}^2+\|\bar{\theta}_1-\bar{\theta}_2\|_{H^s}^2 &\leq C(\|\bar{\omega}_1^0-\bar{\omega}_2^0\|_{H^s}^2+\|\bar{\theta}_1^0-\bar{\theta}_2^0\|_{H^s}^2)\\ &\quad\times e^{\int_0^t(\|\nabla\bar{\omega}_1\|_2^{\frac{1}{2}}\|\Delta\bar{\omega}_1\|_2^{\frac{1}{2}}+\|\nabla\bar{\theta}_1\|_2\|\Delta\bar{\theta}\|_2+1)d\tau}\\ &\leq \epsilon. \end{aligned}\tag{23}$$

This completes the proof of Theorem 1.1. □

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