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Completely generalized right primary rings and their extensions

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Abstract. A ring R is said to be a completely generalized right primary ring ($c.g.r.p$ ring) if $a, b \in R$ with $ab = 0$ implies that $a = 0$ or b is nilpotent.

Let now R be a ring and σ an automorphism of R. In this paper we extend the property of a completely generalized right primary ring ($c.g.r.p$ ring) to the skew polynomial ring $R[x; \sigma]$.

Key Words: Ore extension, automorphism, derivation, completely prime ideal
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Introduction

A ring R means an associative ring with identity $1 \neq 0$. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless other wise stated.

This article concerns the study of skew polynomial rings over completely generalized right primary rings ($c.g.r.p$ rings). Recall that a ring R is said to be a $c.g.r.p$ ring if $a; b \in R$ with $ab = 0$ implies that $a = 0$ or b is nilpotent. An ideal I of R is said to be a completely generalized right primary ideal if R/I is a completely generalized right primary ring ([9]).

Completely generalized left primary ($c.g.l.p$) rings and completely generalized left primary ideals are defined in a similar way.

Example (*Example 2.2. of [9]*) Let A and B be simple nil rings which are not nilpotent (for examples of such rings see Smoktunowicz [12]). Then $R = A \oplus B$ is a nil ring, and R is completely $g.r.p$ ($g.l.p$).

We now give a brief about the skew polynomial rings (also known as Ore extensions):

Let R be a ring, σ an automorphisms of R and δ a σ-derivation of R; i.e. $\delta : R \to R$ is an additive mapping satisfying $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$.

For example let σ be an automorphism of a ring R and $\delta : R \to R$ any map.
Let $\phi : R \to M_2(R)$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then ϕ is a ring homomorphism if and only if δ is a σ-derivation of R.

We recall that the Ore extension

$$R[x;\sigma,\delta] = \{f = \textstyle\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i.e. $\sigma(I) = I$) and is also δ-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$.

In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$. If J is an ideal of R such that J is δ-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x;\delta]$ is an ideal of $D(R)$, and we denote it by $D(J)$. In case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. If K is an ideal of R such that K is σ-stable (i.e. $\sigma(K) = K$), then clearly $K[x;\sigma]$ is an ideal of $S(R)$, and we denote it by $S(K)$.

The study of *c.g.r.p* (*c.g.l.p*) rings stems from Lasker-Noether concept of a primary ideal which has been extended to associative, not necessarily commutative rings. The concept of primary ideal in commutative rings has been generalized to a noncommutative setting by several authors, e.g., Barnes [1], Chatters and Hajarnavis [5], and Fuchs [7]. For more details on the concept of primary ideals and primary decomposition, the reader is referred to Noether [11] and Eisenbud [6].

A stronger type of primary decomposition (called transparency) for a right Noetherian ring has been introduced by the author of this paper in [2] as follows:

Definition 1 *A ring R is said to be an irreducible ring if the intersection of any two non-zero ideals of R is non-zero. An ideal I of R is called irreducible if $I = J \cap K$ implies that either $J = I$ or $K = I$. Note that if I is an irreducible ideal of R, then R/I is an irreducible ring.*

Proposition 1 *Let R be a Noetherian ring. Then there exist irreducible ideals I_j, $1 \leq j \leq n$ of R such that $\cap_{j=1}^{n} I_j = 0$.*

Proof. The proof is obvious and we leave the details to the reader.□

Definition 2 *(Definition 1.2 of [3]) A Noetherian ring R is said to be transparent ring if there exist irreducible ideals $I_j, 1 \leq j \leq n$ such that $\cap_{j=1}^{n} I_j = 0$ and each R/I_j has a right artinian quotient ring.*

It can be easily seen that a Noetherian integral domain is a transparent ring, a commutative Noetherian ring is a transparent ring and so is a Noetherian ring having an artinian quotient ring. A fully bounded Noetherian ring is also a transparent ring.

The following result has been proved in Bhat [2] towards the transparency of skew polynomial rings.

Let R be a commutative Noetherian ring and σ an automorphism of R and δ a σ-derivation of R. Then it is known that $S(R)$ and $D(R)$ are transparent. (Bhat [2])

The following result has been proved in Bhat [3].

Theorem 1 *(Theorem (3.4) of [3]) Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R. Then there exists an integer $m \geq 1$ such that the skew polynomial ring $R[x;\alpha,\delta]$ is a transparent ring, where $\sigma^m = \alpha$ and δ is an α-derivation of R such that $\alpha(\delta(a)) = \delta(\alpha(a))$, for all $a \in R$.*

Completely generalized right primary ring:

We now extend the notion of Completely generalized right primary rings to skew polynomial rings, and have the following:

Definition 3 *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R. We say that $O(R) = R[x;\sigma,\delta]$ is an extended completely generalized right primary ring (e.c.g.r.p ring) if for $f(x), g(x) \in O(R)$ (say $f(x) = \sum_{i=0}^{n} x^i a_i$ and $g(x) = \sum_{j=0}^{m} x^j b_j$), $f(x)g(x) = 0$ implies that $f(x) = 0$ or b_j is nilpotent for all j, $0 \leq j \leq m$.*

We prove the following in this direction:

Theorem A: Let R be a *c.g.r.p* (*c.g.l.p*) ring and σ an automorphism of R. Then $S(R) = R[x;\sigma]$ is an *e.c.g.r.p* (*e.c.g.l.p*) ring. This is proved in Theorem (2).

1 Completely generalized right primary rings and their extensions

We begin this section with the following:

Recall that an ideal P of a ring R is completely prime if R/P is a domain, i.e. $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$ (McCoy [10]).

In commutative case completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

Example 1 *Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$*. If* p *is a prime number, then the ideal* $P = M_2(p\mathbb{Z})$ *is a prime ideal of* R*, but is not completely prime, since for* $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$*, we have* $ab \in P$*, even though* $a \notin P$ *and* $b \notin P$*.*

Towards the completely prime ideals of $O(R)$, the following has been proved in [4]:

Theorem 2 *(Theorem 2.4 of Bhat[4]:) Let* R *be a ring,* σ *an automorphism of* R *and* δ *a* σ*-derivation of* R*. Then:*

1. *For any completely prime ideal* P *of* R *with* $\delta(P) \subseteq P$ *and* $\sigma(P) = P$*,* $O(P)$ *is a completely prime ideal of* $O(R)$*.*

2. *For any completely prime ideal* U *of* $O(R)$*,* $U \cap R$ *is a completely prime ideal of* R*.*

Recall that an ideal I of a ring R is said to be completely semiprime if $a \in R$ such that $a^n \in I$ for some $n \in N$ implies that $a \in I$ (McCoy [10]). With this we have the following known results:

Proposition 2 *(Proposition 2.8. of [9]) Let* I *be a completely semiprime ideal of* R*. Then* I *is a completely prime ideal if and only if* I *is a c.g.r.p (c.g.l.p) ideal of* R*.*

Proposition 3 *(Proposition 2.11. of [9]) Let* $\phi : R \to R'$ *be a surjective homomorphism and* I *an ideal of* R *such that* $Ker(\phi) \subseteq I$*. Then* I *is a c.g.r.p (c.g.l.p) ideal of* R *implies that* $\phi(I)$ *is a c.g.r.p (c.g.l.p) ideal of* R'*.*

Proposition 4 *(Proposition 2.12. of [9]) Let* $\phi : R \to R'$ *be a surjective homomorphism and* I' *an ideal of* R' *with* $I' = \phi^{-1}(I)$*. Then* I' *is c.g.r.p (c.g.l.p) in* R' *implies* I *is c.g.r.p (c.g.l.p) in* R*.*

We now state and prove the main theorem of this article (regarding extended *c.g.r.p* rings) as follows:

Theorem 3 *Let* R *be a c.g.r.p (c.g.l.p) ring and* σ *an automorphism of* R*. Then* $S(R) = R[x; \sigma]$ *is an e.c.g.r.p (e.c.g.l.p) ring.*

Proof. We consider $c.g.r.p$ case. The $c.g.l.p$ shall follow on same lines.

Let $f(x); g(x) \in S(R)$ be such that $f(x)g(x) = 0$ (say $f(x) = \sum_{i=0}^{n} x^i a_i$, $g(x) = \sum_{i=0}^{m} x^i b_i$). We use induction on m, n to prove the result. Let $m = n = 1$ say $f(x) = xa + b$, $g(x) = xc + d$.

Now $f(x)g(x) = 0$ implies that

$$x^2\sigma(a)c + x(\sigma(b)c + ad) + bd = 0$$

This implies that

$$\sigma(a)c = 0, \sigma(b)c + ad = 0, bd = 0$$

Now $bd = 0$ implies that $b = 0$ or d is nilpotent.

Now two cases arises:

1. $b = 0$

2. $b \neq 0$

(1) If $b = 0$, then $\sigma(b)c+ad = 0$ implies that $ad = 0$. Now $ad = 0$ implies that $a = 0$ or d is nilpotent. If $a = 0$, then we have $f(x) = xa + b = 0$. If $a \neq 0$, then d is nilpotent and $\sigma(a) \neq 0$. Therefore $\sigma(a)c = 0$ implies that c is nilpotent. So we have c, d are nilpotent.

(2) If $b \neq 0$, then d is nilpotent. Now $\sigma(a)c = 0$ implies that $\sigma(a) = 0$ or c is nilpotent. If c is nilpotent, we have c, d are nilpotent. If c is non-nilpotent, then $\sigma(a) = 0$ or $a = 0$. Now $\sigma(b)c+ad = 0$ implies that $\sigma(b)c = 0$ and c is non-nilpotent implies that $\sigma(b) = 0$ or $b = 0$. So $f(x) = xa + b = 0$.

Therefore, the result is true for $m = n = 1$.

Suppose the result is true for all polynomials $f(x); g(x)$ with $\deg(f(x)) = n$ and $\deg(g(x)) = m$.

We prove for $f(x); g(x)$ with $\deg(f(x)) = n + 1$ and $\deg(g(x)) = m + 1$. Let

$$f(x) = x^{n+1}c_{n+1} + ... + c_0,\ g(x) = x^{m+1}d_{m+1} + ... + d_0.$$

Now $f(x)g(x) = 0$ implies that

$$x^{m+n+2}\sigma^{m+1}(c_{n+1})d_{m+1} + x^{m+n+1}(\sigma^m(c_{n+1})d_m + \sigma^{m+1}(c_n)d_{m+1}) + ... + c_0d_0 = 0.$$

Now $\sigma^{m+1}(c_{n+1})d_{m+1} = 0$ implies that $\sigma^{m+1}(c_{n+1}) = 0$ or d_{m+1} is nilpotent. Suppose d_{m+1} is non-nilpotent, then $\sigma^m(c_{n+1}) = 0$ or $c_{n+1} = 0$. Also equating coefficient of x^{m+n+1} to zero, we have $\sigma^m(c_{n+1})d_m + \sigma^{m+1}(c_n)d_{m+1} = 0$. Now $c_{n+1} = 0$ implies that $\sigma^{m+1}(c_n)d_{m+1} = 0$ and d_{m+1} is non-nilpotent implies that $\sigma^{m+1}(c_n) = 0$ or $c_n = 0$.

Now equating coefficient of x^{m+n} to zero, we get

$$\sigma^{m-1}(c_{n+1})d_{m-1} + \sigma^m(c_n)d_m + \sigma^{m+1}(c_{n-1})d_{m+1} = 0.$$

Now $c_{n+1} = c_n = 0$ implies that $\sigma^{m+1}(c_{n-1})d_{m+1} = 0$ and d_{m+1} is non-nilpotent implies that $\sigma^{m+1}(c_{n-1}) = 0$ or $c_{n-1} = 0$. With the same process in a finite number of steps we get $c_i = 0$; $0 \leq i \leq n+1$. Therefore, $f(x) = 0$. □

Remark 1 *We have not been able to prove the result for* $O(R) = R[x; \sigma, \delta]$*, where* σ *is an automorphism of* R *and* δ *is a* σ*-derivation of* R*.*

Let $f(x) = xa + b$*,* $g(x) = xc + d$*.*

Now $f(x)g(x) = 0$ *implies that*

$$x^2\sigma(a)c + x(\delta(a)c + \sigma(b)c + ad) + \delta(b)c + bd = 0.$$

So we have

$$\sigma(a)c = 0,\ \delta(a)c + \sigma(b)c + ad = 0,\ \delta(b)c + bd = 0$$

Now $\sigma(a)c = 0$ *implies that* $\sigma(a) = 0$ *or* c *is nilpotent.*

If $\sigma(a) = 0$*, i.e.* $a = 0$*, then* $\delta(a)c+\sigma(b)c+ad = 0$ *implies that* $\sigma(b)c = 0$*. Therefore* $\sigma(b) = 0$ *or* c *is nilpotent. If* $\sigma(b) = 0$*, i.e.* $b = 0$*, then we have* $f(x) = 0$*.*

If $\sigma(b) \neq 0$*, then* c *is nilpotent and* $\delta(b)c + bd = 0$ *gives nothing about the nilpotency of* d*.*

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