PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2018, **52**(3), p. 157–160

Mathematics

ABOUT A CLASS OF THREE-DIMENSIONAL SUBMANIFOLDS IN AFFINE SPACE A^6

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Three-dimensional submanifolds in affine space A^6 have been studied by the method of exterior forms. It is proved that the structure of total space induces a special type of affine connection on this submanifold. The structure equations of this submanifold have been found.

MSC2010: Primary 58A05; Secondary 32B15.

Keywords: affine connection, manifold, three-dimensional submanifold, linear differential forms, the method of exterior forms.

Introduction. Submanifolds in affine space are relatively less studied; only some special cases have been considered. Compared to the geometry of Euclidean space, the moderation of this space they accomplish by so-called equipments [1] of various types or special choice of affine frame. Examples of studies of affine space submanifolds by the method of exterior forms can be found in works [2–5].

There is no any metrics in affine space, so to simplify structure of derivative equations various tools are introduced. If we have three-dimensional submanifold M in affine space A^6, then naturally two bundles appear: tangent and normal. In the tangent bundle there are acting differential forms $\omega^i, \omega^i_k,\ i,\ k = 1,2,3$, and in the normal bundle $\omega^\alpha, \omega^\alpha_\beta,\ \alpha,\ \beta = 4,\ 5,\ 6$. Linear differential forms $\omega^\alpha_i, \omega^i_\alpha,\ i = 1,2,3$ and $\alpha = 4,5,6$, establish connection between tangent and normal bundles. If these forms simultaneously are equal to zero, then the whole space turns into cross product of tangent and normal bundles. In practice, these differential forms are expressed by basic forms, which are used to form the curvature and torsion tensors components of a submanifold.

Let us consider structure equations of affine space A^6, in general case they are given in the following form [6]:

$$\begin{aligned} d\omega^I &= \omega^I_K \wedge \omega^K, \\ d\omega^I_K &= \omega^I_P \wedge \omega^P_K, \quad I,K,P = 1,\ldots,6\ , \end{aligned} \tag{1}$$

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where only the principal forms $\omega^1, \omega^2, \omega^3$ and secondary forms ω_i^k directly describe the structure of submanifold $M \subset A^6$.

It means that our nearest goal is to exclude the rest of principal and secondary forms or to express them through these forms. In submanifold M the forms $\omega^4, \omega^5, \omega^6$ are expressed by the forms $\omega^1, \omega^2, \omega^3$:

$$\omega^\alpha = a_i^\alpha \omega^i, \quad \text{where} \quad i = 1,2,3, \quad \alpha = 4,5,6. \tag{2}$$

These relations are identities on three-dimensional submanifold $M \subset A^6$. It means that the exterior differentiation of these relations gives identities on this submanifold. If we will differentiate these identities in an exterior way and apply structure equations (1), we will get:

$$\begin{aligned}
d\omega^i &= \omega_k^i \wedge \omega^k + \omega_\alpha^i \wedge \omega^{\alpha-3},\\
d\omega^\alpha &= \omega_\beta^\alpha \wedge \omega^\beta + a_p^\alpha a_i^\gamma \omega_\gamma^p \wedge \omega^i,\\
d\omega_k^i &= \omega_p^i \wedge \omega_k^p + \omega_\beta^i \wedge \omega_k^\beta,\\
d\omega_\beta^\alpha &= \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \omega_k^\alpha \wedge \omega_\beta^k,\\
d\omega_i^\alpha &= \omega_k^\alpha \wedge \omega_i^k + \omega_\beta^\alpha \wedge \omega_i^\beta,\\
d\omega_\alpha^i &= \omega_k^i \wedge \omega_\alpha^k + \omega_\beta^i \wedge \omega_\alpha^\beta,
\end{aligned} \tag{3}$$

then after some identical modifications and with further application of Cartan's lemma, we will obtain:

$$da_k^\alpha = a_k^\beta \omega_\beta^\alpha - a_i^\alpha \omega_k^i + \omega_k^\alpha - a_i^\alpha a_k^\beta \omega_\beta^i + a_{kp}^\alpha \omega^p, \tag{4}$$

where the coefficients a_{kp}^α are symmetric with respect to indices k and p, i.e. $a_{kp}^\alpha = a_{pk}^\alpha$. We can assume that in the case of any value of index $\alpha(4,5,6)$, the third order matrix $\left(a_{kp}^\alpha\right)$ is not degenerate: $\det\left(a_{kp}^\alpha\right) \neq 0$. Coefficients a_i^α are components of a relative invariant. It is possible if and only if we have the following condition in the equation (4):

$$\omega_k^\alpha = a_i^\alpha a_k^\beta \omega_\beta^i. \tag{5}$$

For rank of the third order matrix (a_i^α) we have

$$rank\,(a_i^\alpha) = 0,1,2,3.$$

The case of $rank\,(a_i^\alpha) = 1$ has been studied in [5]. It has been indicated that the affine connection of total space generates a special type of affine connection on three-dimensional submanifold in a natural way and the characteristic properties of this connection had been studied.

In the present work the case of $rank\,(a_i^\alpha) = 3$ is observing. Thanks to the canonization of the moving frame, we consider the case of equation (2), in which we have: $a_1^4 = 1,\ a_2^5 = 1,\ a_3^6 = 1$ and all other coefficients are zeros.

It means that equation (2) can be written in the following form:

$$\omega^{\alpha} = \omega^{\alpha-3}. \tag{6}$$

In other words, $a_i^{\alpha} = \delta_i^{\alpha-3}$. If we will substitute this expression in relation (4), then we will obtain

$$0 = \delta_i^{\beta-3}\omega_{\beta}^{\alpha} - \delta_i^{\alpha-3}\omega_k^i + \omega_k^{\alpha} - \delta_i^{\alpha-3}\delta_i^{\beta-3}\omega_{\beta}^i + a_{kp}^{\alpha}\omega^p.$$

Now taking into account the relation (5), we obtain: $\omega_{k+3}^{\alpha} = \omega_k^{\alpha-3} + a_{kp}^{\alpha}\omega^p$.

So the secondary forms ω_{β}^{α} of the normal bundle of submanifold M are expressed by the secondary forms ω_k^i of the tangent bundle of this submanifold. Taking into account the relation (5), we will need to show that the linear differential forms ω_{α}^i are expressed by linear combinations of principal basic forms $\omega^1, \omega^2, \omega^3$. Therefore, the exterior differentiation of the relation (5), which is an identity on submanifold M, and the application of the structure equations of this submanifold and relation (5), we'll obtain the following identity:

$$\left(a_{kp}^{\alpha}a_i^{\beta} + a_k^{\alpha}a_{ip}^{z}\right)\omega_{\beta}^k \wedge \omega^p = 0.$$

According to Cartan's lemma, the linear differential form $\left(a_{kp}^{\alpha}a_i^{\beta} + a_k^{\alpha}a_{ip}^{z}\right)\omega_{\beta}^k$ is expressed by the linear combination of principal basic forms. Now considering the condition $a_i^{\alpha} = \delta_i^{\alpha-3}$ and nondegeneracy of the third order matrix $\left(a_{kp}^{\alpha}\right)$ for any fixed value of the index α, we obtain that the linear differential forms ω_{α}^i are principal forms. This condition can be represented by the following way: $\omega_{k+3}^i = b_{k+3}^i\omega^p$. This, in its turn means that the form $\omega_k^{\alpha} = a_i^{\alpha}a_k^{\beta}\omega_{\beta}^i$ is decomposable by principal basic forms $\omega^1, \omega^2, \omega^3$. Hence, the structure equations of the submanifold M are represented in the following way:

$$\begin{aligned} d\omega^i &= \omega_k^i \wedge \omega^k + a_t^{k+3}b_{kp}^i\omega^p \wedge \omega^t, \\ d\omega_k^i &= \omega_p^i \wedge \omega_k^p + b_{pt}^i b_{qm}^r a_q^{p+3} a_k^{q+3}\omega^t \wedge \omega^m, \end{aligned} \tag{7}$$

where the coefficients b_{kp}^i satisfy the following differential equations:

$$db_{kp}^i = b_{kp}^t\omega_t^i - b_{tp}^i\omega_k^t - b_{kt}^i\omega_p^t + b_{kpt}^i\omega^t, \;\; b_{kpt}^i = b_{kr}^i b_{qp}^r a_t^{q+3}. \tag{8}$$

To obtain these equations it is enough to differentiate the relation $\omega_{k+3}^i = b_{kp}^i\omega^p$ in an exterior way, to apply the structure equations of the submanifold M and apply Cartan's lemma.

Now, substituting the relation $a_i^{\alpha} = \delta_i^{\alpha-3}$ in the obtained equations, it is not difficult to check that the system of differential forms ω^i, ω_k^i, $i,k = 1,2,3$, and functions b_{kp}^i satisfying the system of equations (7), (8), is closed with respect to exterior differentiation, namely the exterior differentiation of these relations does not generate new relations. According to Cartan–Laptev's theorem [7], there exists an affine connection on the submanifold M determined by this system.

Theorem. If a three-dimensional submanifold M in affine space A^6 is given by imbedding (5) and $rank\,(a_i^\alpha) = 3$, then the connection of the total space induces a special type of affine connection on this submanifold, determined by the system of differential forms $\omega^i, \omega_k^i,\ \ i,k = 1,2,3$, and functions b_{kp}^i, satisfying the following differential equations:

$$d\omega^i = \omega_k^i \wedge \omega^k,$$
$$d\omega_k^i = \omega_p^i \wedge \omega_k^p + b_{pt}^i b_{km}^p \omega^t \wedge \omega^m,$$
$$db_{kp}^i = b_{kp}^t \omega_t^i - b_{tp}^i \omega_{k+3}^{t+3} - b_{kt}^i \omega_p^t + b_{kpt}^i \omega^t,\ \ b_{kpt}^i = b_{kr}^i b_{qp}^r.$$

It has a zero torsion and non-zero curvature:

$$R_{tkm}^i = b_{pt}^i b_{km}^p.$$

It follows from Whitney's theorem [8] that any three-dimensional manifold is possible to imbed in the 7 dimensional affine space. The statement of the theorem above shows that in the case $rank\,(a_i^\alpha) = 3$ three-dimensional manifold is admiting an imbedding even in affine space A^6, but under condition of a vanishing torsion. On this matter, it's interesting to study a three-dimensional submanifold M given by structure equations

$$d\omega^i = \omega_k^i \wedge \omega^k + b_{kp}^i \omega^t \wedge \omega^p,$$
$$d\omega_k^i = \omega_p^i \wedge \omega_k^p + b_{pt}^i b_{km}^p \omega^t \wedge \omega^m,$$

where the coefficients b_{pt}^i are not symmetric by lower indices. It's visible that components of its curvature tensor are expressed by the components of the torsion tensor of this affine connection.

This result can be obtained by importing so-called a polarity condition on M (see [1,2]), but that condition limits the class of affine connections and imports affine connections of special type on M.

Received 18.09.2018

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