M a t h e m a t i c s

ON PALETTE INDEX OF UNICYCLE AND BICYCLE GRAPHS

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Given a proper edge coloring ϕ of a graph G, we define the palette $S_G(v, \phi)$ of a vertex $v \in V(G)$ as the set of all colors appearing on edges incident with v. The palette index $\check{s}(G)$ of G is the minimum number of distinct palettes occurring in a proper edge coloring of G. In this paper we give an upper bound on the palette index of a graph G in terms of cyclomatic number $cyc(G)$ of G and maximum degree $\Delta(G)$ of G. We also give a sharp upper bound for the palette index of unicycle and bicycle graphs.

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Introduction. All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. The degree of a vertex v in G is denoted by $d(v)$, and the maximum degree of vertices in G by $\Delta(G)$. The terms and concepts that we do not define can be found in [1].

The problems of coloring the vertices or edges of a graph has always attracted researcher's attention. These problems become interesting and challenging when one requires that the coloring satisfies certain conditions. For instance, the vertices or edges of the graph have to be properly colored, that is, adjacent vertices or adjacent edges have to receive distinct colors. Probably, the most important chromatic parameters for a proper vertex-coloring or edge-coloring in a graph G are the chromatic number of G and the chromatic index of G, denoted by $\chi(G)$ and $\chi'(G)$ respectively. Nevertheless there are many other chromatic parameters. For instance, the circular chromatic index [2], the list chromatic number [3], etc. The chromatic parameters can give some additional and useful information on the structure of a graph. As an example, the chromatic index tells us, if a regular graph is decomposable into perfect matchings.

In this paper we consider a chromatic parameter called the palette index of a simple graph G. This parameter was introduced in [4] and denoted by $\check{s}(G)$. It

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can be defined as follows. Let ϕ be a proper edge-coloring of G and let v be a vertex of G. The set of colors assigned by ϕ to the edges incident to v is called the palette of v (with respect to ϕ) and is denoted by $S_G(v, \phi)$. For every proper edge-coloring of G we can consider the set $S_G(\phi)$ of distinct palettes of ϕ, namely, $S_G(\phi) = \{S_G(v, \phi) \mid v \in V(G)\}$. The cardinality of $S_G(\phi)$ is at most $|V(G)|$. The palette index of G is the minimum number of distinct palettes taken over all proper edge-colorings of G, namely, $\check{s}(G) = \min\{|S_G(\phi)| \mid \phi \text{ proper edge coloring of } G\}$.

As shown in [4], the palette index of a regular graph is 1 if and only if the graph is of Class 1 (see [5] for the definition of graphs of Class 1 and 2). Moreover, it is different from 2. As mentioned in [6], the palette index of a d-regular graph of Class 2 satisfies the inequality $3 \le \check{s}(G) \le d + 1$. Hence, $\check{s}(G) = 3$, if G is a 2-regular graph of Class 2.

In [4] it was studied the palette index of cubic graphs. More specifically,

$$\check{s}(G) = \begin{cases} 1, & \text{if } G \text{ is of Class 1;} \\ 3, & \text{if } G \text{ is of Class 2 and has a perfect matching;} \\ 4, & \text{if } G \text{ is of Class 2 and has no perfect matching.} \end{cases}$$

In [4] the palette index of the complete graph K_n, $n \ge 4$, is found. Namely,

$$\check{s}(K_n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 2, \\ 3, & \text{if } n \equiv 3 \pmod 4, \\ 4, & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

The paper [6] investigates 4-regular graphs and proves that $\check{s}(G) \in \{1, 3, 4\}$, if G is 4-regular and of Class 2, and that all these values are in fact attained.

Since the computing of the chromatic index of cubic graphs is *NP*-complete [7], determining the palette index of a given graph is also *NP*-complete, even for cubic graphs [4]. This in fact means that even determining, if a given graph has palette index 1 is an *NP*-complete problem.

Vizing's edge coloring theorem yields an upper bound for the palette index of a general graph G with maximum degree Δ, namely $\check{s}(G) \le 2^{\Delta+1} - 2$. It is not hard to construct graphs, which palette index is quite small than $2^{\Delta+1} - 2$. Indeed, it was described in [8] an infinite family of multigraphs, whose palette index grows asymptotically as Δ^2. It is an open question whether there are such examples without multiple edges. Furthermore, [8] suggests to prove that there is a polynomial $p(\Delta)$, so that for any graph with maximum degree Δ, it holds the bound $\check{s}(G) \le p(\Delta)$.

There are few results about the palette index of non-regular graphs. In [9] it is completely determined the palette index of the complete bipartite graphs $K_{a,b}$ with $a < 5$. It was studied in [10], the palette index of bipartite graphs. In particular, it is determined the exact value of the palette index of grids.

[11] studies the palette index of trees, proving that $\check{s}(T) \le \sum_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil$. The paper also proves the sharpness of the bound constructing T^{Δ} graphs, for which the palette index is $\sum_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil$.

In this paper we focus on non-regular graphs. In particular, we give tight upper bound for the palette index of unicycle graphs. We give tight upper bound for the palette index of bicycle graphs. Moreover, we derive a general upper bound for the palette index of graphs in terms of the cyclomatic number and the maximum degree.

Definitions. Let us recall the algorithm for coloring of trees with a small number of palettes. As shown in [11], we can color a tree T with the maximum degree Δ by using following palettes:

$$P_j^i = \{ij, ij+1, ij+2, \ldots, ij+i-1\}$$

of cardinality i, where $i = 1, \ldots, \Delta$ and $j = 0, \ldots, \left[\frac{\Delta}{i}\right] - 1$ and all colors are taken modulo Δ. For example, for a tree with the maximum degree 5 we have the following palettes:

$\{0\}, \{1\}, \{2\}, \{3\}, \{4\}$
$\{0,1\}, \{2,3\}, \{4,0\}$
$\{0,1,2\}, \{3,4,0\}$
$\{0,1,2,3\}, \{4,0,1,2\}$
$\{0,1,2,3,4,5\}$.

We color the tree T in the following way.

Choose one of the vertices of T, say v, to be the root of the tree. Assign arbitrarily to the edges incident with v the colors of the palette $P_0^{d(v)}$. Start a breadth-first search on the vertices of T. Each time we visit a new vertex u of T, exactly one of the edges incident with u, say ux, is already colored. We consider (one of) the sets $P_j^{d(u)}$ containing the color of ux (i.e. this set does exist, since for each i the sets P_j^i cover all colors at least once). We assign the colors of $P_j^{d(u)}$ different from the color of ux to the remaining edges incident with u. It is clear that we can continue this process until all edges of T are colored.

Observe that we need for each color exactly one palette of cardinality $1, \ldots, \Delta$. Which is not possible for all values of Δ and cardinalities of palettes.

D e f i n i t i o n 1. (*Original Palette*). For a given $\Delta \in N$ the set of palettes $\{P_j^i \mid i \in I, j \in J\}$, where $I = \{1, 2, \ldots, \Delta\}$ and $J = \{0, \ldots, \left[\frac{\Delta}{i}\right] - 1\}$, is called the set of original palettes, if for each $i \in I$ and $j \in J$

$$P_j^i = \{ij, ij+1, ij+2, \ldots, ij+i-1\}$$

and all colors are taken modulo Δ.

D e f i n i t i o n 2. (*Standard Palette*). For a given $\Delta \in N$ the set of palettes $\{P_j^i \mid i \in I, j \in J\}$, where $I = \{1, 2, \ldots, \Delta\}$ and $J = \{0, \ldots, \left[\frac{\Delta}{i}\right] - 1\}$, is called the set of standard palettes, if for each $i \in I$, $\bigcup_{j \in J} P_j^i = \{0, 1, \ldots, \Delta - 1\}$, and there is an index $j' \in J$, for which $P_{j_1}^i \cap P_{j_2}^i = \emptyset$, where $j_1, j_2 \in J \setminus \{j'\}$.

D e f i n i t i o n 3. (*Extra Palette*). If we have already defined a set of standard palettes of a given $\Delta \in N$, then any palette different from these is called an extra palette.

From the Definitions it follows that the set of original palettes is also a set of standard palettes.

Palette Index of Unicycle Graphs. Unicycle graph is a connected, simple graph, which owns only one cycle.

D e f i n i t i o n 4. (*Condition* *). We will say the unicycle graph G with the cycle C satisfies Condition *, if:

- the cycle C has an odd length;
- there is a vertex with a degree greater than 2, which belongs to $V(C)$;
- there are no 2 neighbor vertices from $V(C)$ with a degree greater than 2;
- the maximum degree of G is even.

T h e o r e m 1. For any unicycle graph G the following inequality is true:

$$\check{s}(G) \leq \begin{cases} \sum_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil + 1, & \text{if } G \text{ satisfies Condition *,} \\ \sum_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil & \text{otherwise.} \end{cases}$$

Moreover, this upper bound is sharp.

P r o o f. Let C be the cycle from the graph G.

As an initial set of palettes, we use the set of original palettes for Δ.

It is easy to see the inequality is true when G is a cycle, since we can color a cycle using the set of palettes $\{\{0,1\}\}$ if the cycle is even, and the set of palettes $\{\{0,1\},\{0,2\},\{1,2\}\}$ if the cycle is odd. Since $\sum_{i=1}^{2} \left\lceil \dfrac{2}{i} \right\rceil = 3$ and we can color G by using at most 3 distinct palettes, the inequality is true.

Now we consider the case when C contains at least 1 vertex with a degree greater than 2. First, we color cycle C. Then, we color other parts of graph G.

Let $v \in V(C)$ be a vertex with a degree greater than 2, and suppose that the edges av and bv belong to $E(C)$. Let us call a v-subtree of the graph G the connected part of the graph $G - (V(C) \setminus \{a,v,b\})$ containing the vertex v, and let us denote it by T_v.

Case 1. The Length of C is Even.

We color C with the colors 0 and 1. For each vertex $u \in V(C)$ we have two adjacent edges colored by the colors 0 and 1. The set of original palettes contains palettes of cardinality $2,\ldots,\Delta$, which contain the colors 0 and 1. So we can color the subtree T_u using palettes from the set of original palettes. Consequently, we use at most $\sum_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil$ palettes.

Case 2. The Length of C is Odd.

Suppose there are no two neighbor vertices on the cycle C with a degree greater than 2. Let the vertex v belong to $V(C)$ and $d(v) > 2$, and suppose that vertices $s \in V(C)$ and $t \in V(C)$, both of a degree 2, are neighbors of the vertex v. We color sv by the color 1, tv by the color 2. The odd path $C - v$ we color consecutively by the colors 0, 1 starting from the vertex s. Since we can color

uncolored edges adjacent to v by the colors $\{0,3,4,\ldots,\Delta-1\}$, we can color the subtree T_v as a tree. Then, if Δ is odd we replace the palette $\{0,\Delta-1\}$ from the original palette set with the palette $\{0,2\}$.

For each other vertex $u \in V(C) \setminus \{v\}$ with $d(u) > 2$ we have a subtree T_u and we can color it as a tree. It is easy to see that we use one extra palette $\{0,2\}$ if Δ is even.

If there are neighbor vertices $u,v \in V(C)$ with a degree greater than 2, then we color uv by 2 and we color the even path $C - uv$ consecutively by the colors 0 and 1 starting from the vertex v. Since we can color uncolored edges adjacent to v by the colors $\{1,3,4,\ldots,\Delta-1\}$, we can color the subtree T_v as a tree. Since we can color uncolored edges adjacent to u by the colors $\{1,3,4,\ldots,\Delta-1\}$, we can color the subtree T_u as a tree. Then, we color all remaining subtrees. Consequently, we use at most $\sum\limits_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil$ palettes.

Now let us show that the equality is possible. We use the tree T^{Δ} from [11]. Let us take an arbitrary simple cycle S with an odd length, and let us connect an arbitrary vertex of S with any leaf of T^{Δ} with an edge. We can not color the cycle S using less than 3 palettes. If the equality is not possible, then removing the vertices of the cycle S we get a coloring of T^{Δ} that uses less than $\sum\limits_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil$ palettes, which is impossible. $\qquad\square$

Palette Index of Bicycle Graphs. Now we consider connected, simple graphs having cyclomatic number 2. We call them bicycle graphs. We give a tight upper bound for the palette index of bicycle graphs.

We divide bicycle graphs into three groups:

1) there are 2 vertices u and v connected by 3 disjoint paths: l_1, l_2, and l_3;

2) there are 2 cycles C_1 and C_2 with the unique common vertex v;

3) there are 2 cycles C_1 and C_2 connected by a path l.

It is not difficult to see that an arbitrary bicycle graph belongs to one of this groups.

T h e o r e m 2. For any bicycle graph G the following inequality is true

$$\check{s}(G) \le \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 2,$$

and this bound is sharp.

P r o o f. We give a coloring for a bicycle graph G using maximum $\sum\limits_{i=1}^{\Delta} \left\lceil \dfrac{\Delta}{i} \right\rceil + 2$ palettes. First we color cycles, then other parts of G. As an initial set of palettes we use the set of original palettes of Δ. If there is a need, we will replace some palettes and will keep the set of palettes as a set of standard palettes.

If *G* is from the First Group.
We have 4 cases:

	l_1	l_2	l_3
Case 1	even	even	even
Case 2	even	even	odd
Case 3	even	odd	odd
Case 4	odd	odd	odd

Case 1 and Case 3. We color
- path l_1 consecutively by the colors 2 and 3 starting from u;
- path l_2 consecutively by the colors 0 and 1 starting from u;
- path l_3 consecutively by the colors 1 and 0 starting from u.

So we have palettes $\{0,1\}$, $\{2,3\}$, $\{0,1,2\}$ and $\{0,1,3\}$. Thus we already have 1 extra palette $\{0,1,3\}$, because the colors 0 and 1 already exist in the palette $\{0,1,2\}$. When $\Delta = 3$, we replace the palette $\{0,2\}$ from the set of original palettes with the palette $\{2,3\}$. Let $x \in V(l_1) \setminus \{u,v\}$. If $d(x) = 3$, then we have one more extra palette, because we have the color 2 in the palette $\{0,1,2\}$. If $d(x) > 3$, then we can color the uncolored edges adjacent to x using a palette from the set of original palettes. It is easy to see that all uncolored parts of G we can color as trees. Consequently, we have at most $\sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 2$ distinct palettes.

Case 2 and Case 4. We color
- path l_1 consecutively by the colors 0 and 1 starting from u;
- path l_2 consecutively by the colors 1 and 0 starting from u;
- path l_3 consecutively by the colors 2 and 3 starting from u.

So we have palettes $\{0,1\}$, $\{2,3\}$ and $\{0,1,2\}$. When $\Delta = 3$, we replace the palette $\{0,2\}$ from the set of original palettes with the palette $\{2,3\}$. Let $x \in V(l_3) \setminus \{u,v\}$. If $d(x) = 3$, then we have one extra palette, because we have 2 in the palette $\{0,1,2\}$. If $d(x) > 3$, then we can color the uncolored edges adjacent to x using a palette from the set of original palettes. It is easy to see that we can color all uncolored parts of G as trees. Consequently, we have at most $\sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 1$ distinct palettes.

If *G* is from the Second Group.
Three cases are possible here:

	C_1	C_2
Case 1	even	even
Case 2	even	odd
Case 3	odd	odd

Case 1.
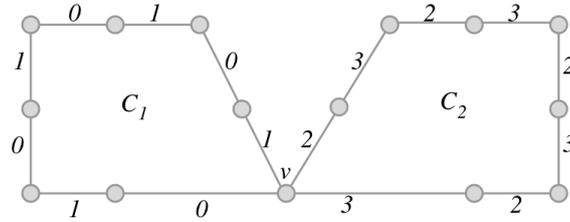
We color C_1 by the colors 0 and 1 and C_2 by the colors 2 and 3. So we have palettes $\{0,1\}$, $\{2,3\}$, $\{0,1,2,3\}$. If there is an uncolored edge e adjacent to a vertex $u \in (V(C_1) \cup V(C_2)) \setminus \{v\}$, then we can color e without using an extra palette of the cardinality 3, because $\{0,1\} \cap \{2,3\} = \emptyset$. If there are 2 or more uncolored edges adjacent to a vertex $u \in (V(C_1) \cup V(C_2)) \setminus \{v\}$ or one or more uncolored edges adjacent to the vertex v, then we can use an interval starting from 0. It is easy to see that all uncolored parts of G can be colored as trees. It means we can color G using at most $\sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil$ distinct palettes.

Case 2.
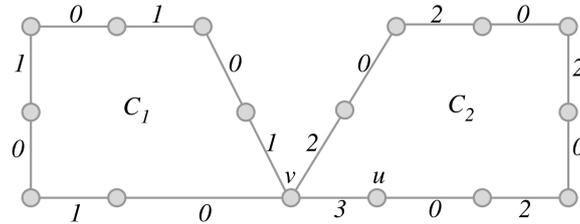
We color the even cycle C_1 by the colors 0 and 1 and the odd cycle C_2 by the colors $2, 0, ..., 2, 0, 3$ starting from the vertex v. So we have the palettes $\{0,1\}$, $\{0,2\}$, $\{0,3\}$, $\{0,1,2,3\}$. Since the color 0 appears in 3 palettes with the cardinality 2, we have at most 2 extra palettes. Let us notice that we have only one vertex u colored by $\{0,3\}$. If there is an uncolored edge adjacent to u, then we may use an extra palette of cardinality 3, but we do not have the palette $\{0,3\}$. If there is an uncolored edge e' adjacent to a vertex $x \in (V(C_1) \cup V(C_2)) \setminus \{v,u\}$, then we can color e' by the color 2 or 1, and use the palette $\{0,1,2\}$. If there are 2 or more uncolored edges adjacent to a vertex $x \in (V(C_1) \cup V(C_2)) \setminus \{v\}$ or one or more uncolored edges adjacent to the vertex v, then we can make an interval starting from 0, which is from the set of standard palettes. It is easy to see that all uncolored parts of G can be colored as trees. It means we can color G using at most $\sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 2$ palettes.

Case 3.
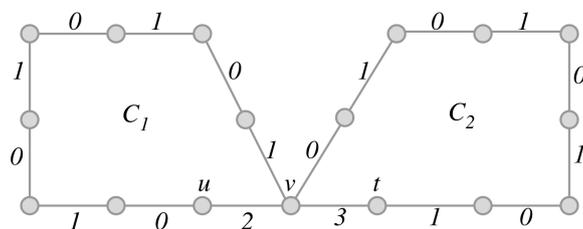
We color C_1 by colors $1, 0, \ldots, 1, 0, 2$ starting from the vertex v and C_2 by colors $0, 1, \ldots, 0, 1, 3$ starting from the vertex v. So we have the palettes $\{0, 1\}$, $\{0, 2\}$, $\{1, 3\}$, $\{0, 1, 2, 3\}$, and only the vertex u has the palette $\{0, 2\}$, and the vertex t has the palette $\{1, 3\}$. If there is no uncolored edge adjacent to a vertex from $(V(C_1) \cup V(C_2)) \setminus \{v\}$, then we may have 2 extra palettes ($\{0, 2\}$ and $\{1, 3\}$). If there are 2 or more uncolored edges adjacent to a vertex $x \in (V(C_1) \cup V(C_2)) \setminus \{v\}$ or one or more uncolored edges adjacent to the vertex v, then we can use an interval starting from 0. If there is an uncolored edge adjacent to the vertex $x \in (V(C_1) \cup V(C_2)) \setminus \{v, u, t\}$, then we color it by the color 2. If there is an uncolored edge adjacent to the vertex u, then we color it by 1. If there is an uncolored edge adjacent to the vertex t, then we may construct a new palette, but we remove the palette $\{1, 3\}$. It is easy to see that we can color all uncolored parts of G as trees. It means we can color G using at most

$$\sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 2 \text{ distinct palettes.}$$

If G is from the Third Group.

Two cases are possible: the sum M of the lengths of C_1, C_2, and l is even or odd.

Case I   *Case II*
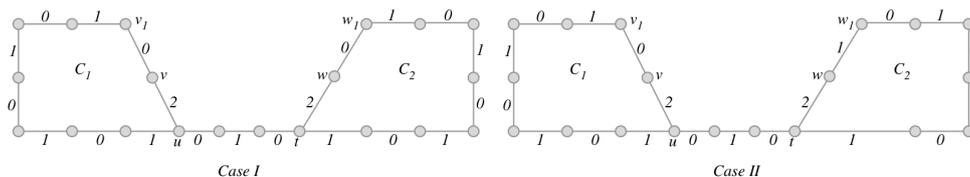
We color uv and tw by the color 2, and the path $v, v_1, \ldots, u, \ldots, t, \ldots, w_1, w$ with colors 0 and 1. So we have the following palettes:

1. $\{0, 1, 2\}$, $\{0, 1\}$, $\{0, 2\}$, if M is odd;
2. $\{0, 1, 2\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$, if M is even.

In the first case we have 1 extra palette $\{0, 2\}$. If there are one or more uncolored edges adjacent to a vertex $x \in V(C_1) \cup V(C_2) \cup V(l)$, then we can color these edges using an interval starting from 0, which is from the set of original palettes. It is easy to see that all uncolored parts of G can be colored as trees. So we can color the graph G using at most $\sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 1$ distinct palettes.

In the second case, we have 2 extra palettes: $\{0, 2\}$ and $\{1, 2\}$. If there are one or more uncolored edges adjacent to a vertex $x \in V(C_1) \cup V(C_2) \cup V(l)$, then

we can color these edges using an interval starting from 0, which is from the set of original palettes. It is easy to see that we can color all uncolored parts of G as trees. So we can color the graph G using at most $\sum_{i=1}^{\Delta} \left[\dfrac{\Delta}{i}\right] + 2$ distinct palettes.

Now let us show that this bound is tight. Assume the contrary:

$$\check{s}(G) < \sum_{i=1}^{\Delta} \left[\frac{\Delta}{i}\right] + 2.$$

Set $\Delta = 6k$, where $k = 1, 2, \ldots$
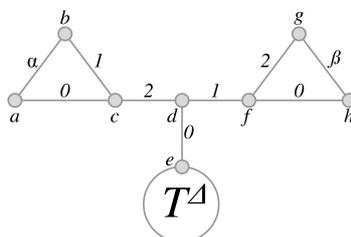
We use the tree T^{Δ} from [11]. For coloring of the tree T^{Δ} we use $\sum_{i=1}^{\Delta} \left[\dfrac{\Delta}{i}\right]$ distinct palettes. Since $\Delta = 6k$, among these palettes there are no two different palettes P_i and P_j with the same cardinality of 2 or 3, with non empty intersection.

Let us color edges adjacent to the vertex c. For example, we color ca by 0, cb by 1 and cd by 2. Let us color the edge ab by the color α. So we have the palettes $\{0, \alpha\}$, $\{1, \alpha\}$, $\{0, 1, 2\}$. So we already have 1 extra palette, because the color α appears in two palettes of cardinality 2. Since $\Delta = 6k$, the degree of the vertex d is 3, and cd has the color 2, we must use the palette $\{0, 1, 2\}$ for the vertex d, otherwise we will have 2 extra palettes. Without loss of generality, we color de by 0 and df by 1. We can say the same about the vertex f. Without loss of generality, we color fg by 2 and fh by 0. Let us color the edge gh by the color β. So we also have the palettes $\{0, \beta\}$, $\{2, \beta\}$. If $\alpha \neq \beta$, then we have 2 extra palettes (second extra palette is $\{0, \beta\}$) so $\alpha = \beta$. But if $\alpha = \beta$, then we have $\{0, \alpha\}$, $\{1, \alpha\}$, $\{2, \alpha\}$. That means we have 2 extra palettes which is a contradiction. □

Palette Index of a Graph in Terms of Cyclomatic Number and Maximum Degree.

T h e o r e m 3. For any graph G, we have

$$\check{s}(G) \leq \sum_{i=1}^{\Delta'} \left[\frac{\Delta'}{i}\right] + 2cyc(G),$$

where $\Delta' = \min\{\Delta(H) \mid H \text{ is a spanning tree of } G\}$.

P r o o f. We color G by 2 phase. First, we color a spanning tree of G, which has the least maximum degree. Then we color edges, which are not colored yet. There are $cyc(G)$ such edges.

We denote by H a spanning tree of G with $\Delta(H) = \Delta'$. We color the tree H as described in [11]. Then we have at most $\sum_{i=1}^{\Delta'} \left\lceil \dfrac{\Delta'}{i} \right\rceil$ distinct palettes.

Then, we have uncolored $L = E(G) \setminus E(H) = \{e_0, \ldots, e_{cyc(G)-1}\}$ edges. Each edge e_i, $i = 0, \ldots, cyc(G) - 1$, can be colored by $\Delta' + i$. It creates 1 or 2 new palettes. Then we have at most $2cyc(G)$ new palettes.

Hence,

$$\check{s}(G) \leq \sum_{i=1}^{\Delta'} \left\lceil \frac{\Delta'}{i} \right\rceil + 2cyc(G).$$

$\square$

C o r o l l a r y . For any graph G we have

$$\check{s}(G) \leq \sum_{i=1}^{\Delta} \left\lceil \frac{\Delta}{i} \right\rceil + 2cyc(G).$$

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