PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2021, **55**(1), p. 20–28

Mathematics

ON WEIGHTED SOLUTIONS OF $\overline{\partial}$-EQUATION IN THE UNIT DISC

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In the paper an equation $\partial g(z)/\partial \bar{z} = v(z)$ is considered in the unit disc $\mathbb{D}$. For C^k-functions v $(k = 1,2,3,\ldots,\infty)$ from weighted L^p-classes $(1 \le p < \infty)$ with weight functions of the type $|z|^{2\gamma}(1-|z|^{2\rho})^{\alpha}$, $z \in \mathbb{D}$, a family g_β of solutions is constructed (β is a complex parameter).

https://doi.org/10.46991/PYSU:A/2021.55.1.020

MSC2010: 32W05, 30H20, 30C40, 30E20.

Keywords: $\overline{\partial}$-equation, weighted function spaces.

Introduction. In [1] a generalization of the famous Cauchy integral formula for smooth functions was presented. More exactly, if Ω is a bounded domain with piecewise smooth boundary and $f \in C^1(\overline{\Omega})$, then the following formula holds (so-called Cauchy–Green formula):

$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint\limits_{\Omega} \frac{\dfrac{\partial f(\zeta)}{\partial \overline{\zeta}}}{\zeta - z} dm(\zeta), \quad z \in \Omega, \tag{1}$$

where m is two-dimensional Lebesgue measure in the complex plane. Recall that

$$\frac{\partial}{\partial \overline{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad (\zeta = x + iy) \tag{2}$$

is the Cauchy–Riemann operator and it annihilates holomorphic functions, i.e. $\dfrac{\partial f(\zeta)}{\partial \overline{\zeta}} \equiv 0$ if f is holomorphic in Ω. As the first summand in (1) is holomorphic in Ω, we can conclude that a solution of so-called $\overline{\partial}$-equation

$$\frac{\partial g(z)}{\partial \bar{z}} = v(z), \quad z \in \Omega, \tag{3}$$

with given $v \in C^1(\Omega)$ and unknown $g \in C^1(\Omega)$ **can be** found in the form

$$g(z) = -\frac{1}{\pi} \iint\limits_{\Omega} \frac{v(\zeta)}{\zeta - z} dm(\zeta), \quad z \in \Omega. \tag{4}$$

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The Eq. (3) plays an important role in complex analysis (especially in several complex variables). Nevertheless, in one complex variable it also had important applications in the corona problem solution and approximation theory.

Recall the following results, concerning the solution of Eq. (3).

Theorem 1. *(Theorem 1.2.2, [2]). If $\Omega \subset \mathbb{C}$ is an open bounded set, $k = 1,2,3,\ldots,\infty$ and $v \in C_c^k(\Omega)$, i.e. $v \in C^k(\Omega)$ and v has a compact support in Ω, then the function g defined by the formula (4) belongs to $C^k(\Omega)$ and satisfies the Eq. (3).*

Theorem 2. *(Proposition 16.3.2, [3]; Theorem 1.1.3, [4]). If $\Omega \subset \mathbb{C}$ is an open bounded set, $k = 1,2,3,\ldots,\infty$ and $v \in C^k(\Omega)\cap L^\infty(\Omega)$ or $v \in C^k(\Omega)\cap L^1(\Omega)$, then the function g defined by the formula (4) belongs to $C^k(\Omega)$ and satisfies the Eq. (3). Moreover, we have*

$$\|g\|_\infty \le (diam(\Omega))^2 \cdot \|v\|_\infty \quad or \quad \|g\|_1 \le (diam(\Omega))^2 \cdot \|v\|_1. \tag{5}$$

In [5], where the cases of the unit ball $B_n \subset \mathbb{C}_n$ and the unit polydisc $U^n \subset \mathbb{C}_n$ were considered, it was given the following generalization of (1) for the unit disc $\mathbb{D} = \{\zeta : |\zeta| < 1\}$ ($Re\beta > -1$ and $f \in C^1(\overline{\mathbb{D}})$):

$$f(z) = \frac{\beta+1}{\pi}\iint\limits_{\mathbb{D}} \frac{f(\zeta)(1-|\zeta|^2)^\beta}{(1-z\overline{\zeta})^{2+\beta}}dm(\zeta)$$

$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{\dfrac{\partial f(\zeta)}{\partial\overline{\zeta}}}{\zeta - z}\left(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D}, \tag{6}$$

where the first summand of (6) is holomorphic in $z \in \mathbb{D}$ that was first appeared in [6,7]. Hence, similar to (4), the second summand of (6) can serve as a formula for a solution of $\overline{\partial}$-equation (3):

$$g_\beta(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{v(\zeta)}{\zeta - z}\left(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\right)^{\beta+1} dm(\zeta), \quad z \in \mathbb{D}. \tag{7}$$

Namely, the following assertion holds:

Theorem 3. *Assume that $1 \le p < +\infty$, $\alpha > -1$ and $Re\beta > \alpha$. If $v \in C^1(\mathbb{D}) \cap L^p_{\alpha+1}(\mathbb{D})$, then the function g_β defined by the formula (7) belongs to $C^1(\mathbb{D}) \cap L^p_\alpha(\mathbb{D})$ and satisfies the equation (3). Moreover, we have*

$$\|g_\beta\|_{p,\alpha} \le const(\alpha,\beta)\|v\|_{p,\alpha+1}. \tag{8}$$

This Theorem is a consequence of a corresponding multidimensional result of [5]. In its formulation the following notations are used:

$$\|f\|^p_{p,\alpha} = \iint\limits_{\mathbb{D}} |f(\zeta)|^p(1-|\zeta|)^\alpha dm(\zeta),$$

$$L^p_\alpha(\mathbb{D}) = \{f(\zeta), \zeta \in \mathbb{D} : \|f\|_{p,\alpha} < +\infty\}.$$

Note that integral representations of type (6) obtained in [5] for the unit ball B_n, were generalized in [8] (for the matrix unit disc) and in [9] (where very general weight functions were considered for the unit ball B_n).

Further generalizations of the formula (6) for the unit disc $\mathbb{D}$ were obtained in [10–13] (under various assumptions on $f(\zeta)$ and $\partial f(\zeta)/\partial\overline{\zeta}$) and can be written as follows:

$$f(z) = \iint\limits_{\mathbb{D}} f(\zeta) S_{\beta,\rho,\varphi}(z;\zeta)(1-|\zeta|^{2\rho})^{\beta}|\zeta|^{2\varphi} dm(\zeta)$$

$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{\frac{\partial f(\zeta)}{\partial\overline{\zeta}}}{\zeta - z} Q_{\beta,\rho,\varphi}(z;\zeta) dm(\zeta), \qquad z \in \mathbb{D}, \quad (9)$$

where the kernels S and Q were represented in an explicit (integral or series) form.

In the present paper we show (Theorems 4 and 5) that the second part of (9) generates a family of solutions of $\overline{\partial}$-equation (3) in $\mathbb{D}$.

Preliminaries. In this section we present several formulas and facts from [10] and [13].

Assume that $\rho > 0$, $\alpha > -1$ and $\gamma > -1$. For $1 \le p < +\infty$ and arbitrary complex-valued measurable function $f(\zeta)$, $\zeta \in \mathbb{D}$, put

$$M^p_{\alpha,\rho,\gamma}(f) = \iint\limits_{\mathbb{D}} |f(\zeta)|^p (1-|\zeta|^{2\rho})^{\alpha}|\zeta|^{2\gamma} dm(\zeta) \qquad (10)$$

and define

$$L^p_{\alpha,\rho,\gamma}(\mathbb{D}) = \{f(\zeta), \zeta \in \mathbb{D} : M^p_{\alpha,\rho,\gamma}(f) < +\infty\}. \qquad (11)$$

Evidently, $L^p_{\alpha,\rho,\gamma}(\mathbb{D}) \subset L^1_{\alpha,\rho,\gamma}(\mathbb{D}), 1 \le p < +\infty$ (see Proposition 3.3 in [14]).

Assume that $Re\beta > -1$, $Re\varphi > -1$ and $\mu = (\varphi+1)/\rho$. For arbitrary $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}$ the kernel $Q_{\beta,\rho,\varphi}(z;\zeta) \equiv Q(z;\zeta)$ is defined as follows:

$$Q_{\beta,\rho,\varphi}(z;\zeta) = 1 + \frac{(z-\zeta)\rho}{\zeta\,\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \frac{z^k}{\zeta^k} \int\limits_0^{|\zeta|^2} (1-t^{\rho})^{\beta} t^{\varphi+k} dt$$

$$\equiv 1 + \frac{z-\zeta}{\zeta\,\Gamma(\beta+1)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\mu+\beta+1+\frac{k}{\rho}\right)}{\Gamma\left(\mu+\frac{k}{\rho}\right)} \cdot \frac{z^k}{\zeta^k} \int\limits_0^{|\zeta|^{2\rho}} (1-x)^{\beta} x^{\mu+\frac{k}{\rho}-1} dx, \qquad (12)$$

where $\zeta \in \overline{\mathbb{D}} \backslash \{0\}$ and

$$Q_{\beta,\rho,\varphi}(z;0) \equiv 1. \qquad (13)$$

It was proved in the Section 2.2 in [10], that in the special case $\rho = 1$ and $\varphi = 0$ the kernel Q takes the form $\left(\frac{1-|\zeta|^2}{1-z\overline{\zeta}}\right)^{\beta+1}$.

The following assertions describe a part of the properties of the introduced kernel.

Proposition 1. *The kernel* $Q(z;\zeta)$ *is well-defined for* $z\in\mathbb{D}$ *and* $\zeta\in\overline{\mathbb{D}}$ *by* (12) *and* (13). *Moreover,* $Q(z;\zeta)$ *is continuous in* $\overline{\mathbb{D}}\backslash\{0\}$ *for a fixed* z *and holomorphic in* $\mathbb{D}$ *for a fixed* ζ.

Proposition 2. *Suppose* $0<|\zeta|\le\frac{1}{2}$. *Then*

$$|Q(z;\zeta)-Q(z;0)|\equiv|Q(z;\zeta)-1|\le\frac{const(\beta,\rho,\varphi)}{(1-|z|)^{Re\beta+1}}\begin{cases}|\zeta|^{2Re\varphi+1}, & z\neq 0,\\ |\zeta|^{2Re\varphi+2}, & z=0.\end{cases}\tag{14}$$

Proposition 3. *Let* $(1+|z|)/2\le|\zeta|\le 1$, *then*

$$|Q(z;\zeta)|\le const(\beta,\rho,\varphi)\frac{\left(1-|\zeta|^{2\rho}\right)^{Re\beta+1}}{(1-|z|)^{Re\beta+2}}.\tag{15}$$

Weighted Solutions of $\overline{\partial}$-equation in $\mathbb{D}$. Let $\rho>0$, $Re\beta>-1$, $Re\varphi>-1$ and $\mu=(\varphi+1)/\rho$. Assume also that $Q_{\beta,\rho,\varphi}(z;\zeta)\equiv Q(z;\zeta)$ is defined by (12), (13) and for a function $v(\zeta)$, $\zeta\in\mathbb{D}$, put formally

$$g_{\beta,\rho,\varphi}(z)=-\frac{1}{\pi}\iint\limits_{\mathbb{D}}\frac{v(\zeta)}{\zeta-z}Q_{\beta,\rho,\varphi}(z;\zeta)dm(\zeta),\quad z\in\mathbb{D}.\tag{16}$$

Theorem 4. *If* $v\in C_c^k(\mathbb{D})$, $k=1,2,3,\dots,\infty$, *then* $g(z)\equiv g_{\beta,\rho,\varphi}(z)$ *is of class* $C^k(\mathbb{D})$ *and satisfies the* $\overline{\partial}$*-equation* (3).

Proof. Obviously, $|v(\zeta)|\le M$, $\zeta\in\mathbb{D}$. According to Propositions 1 and 2 the integral of (16) is convergent for every $z\in\mathbb{D}$, i.e. the function g is well-defined. Using the formulas (12) and (13), we can write $g(z)$ in an expanded form:

$$g(z)=-\frac{1}{\pi}\iint\limits_{\mathbb{D}}\frac{v(\zeta)}{\zeta-z}dm(\zeta)+\frac{1}{\pi}\cdot\frac{\rho}{\Gamma(\beta+1)}$$
$$\times\iint\limits_{\mathbb{D}}v(\zeta)\sum_{k=0}^{\infty}\left(\frac{\Gamma(\mu+\beta+1+k/\rho)}{\Gamma(\mu+k/\rho)}\cdot\frac{z^k}{\zeta^{k+1}}\int\limits_0^{|\zeta|^2}(1-t^\rho)^\beta t^{\varphi+k}dt\right)dm(\zeta)$$
$$=-\frac{1}{\pi}\iint\limits_{\mathbb{D}}\frac{v(\zeta)}{\zeta-z}dm(\zeta)+\frac{1}{\pi}\cdot\frac{\rho}{\Gamma(\beta+1)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{\Gamma(\mu+\beta+1+k/\rho)}{\Gamma(\mu+k/\rho)}z^k\iint\limits_{\mathbb{D}}\frac{v(\zeta)}{\zeta^{k+1}}\int\limits_0^{|\zeta|^2}(1-t^\rho)^\beta t^{\varphi+k}dt\,dm(\zeta)\right),$$

where the change of the order of the summation and the integration is justified by the following chain of inequalities:

$$\begin{aligned}&\left|\sum_{k=0}^{n}\frac{\Gamma(\mu+\beta+1+k/\rho)}{\Gamma(\mu+k/\rho)}v(\zeta)\frac{z^k}{\zeta^{k+1}}\zeta^{2k}\right|\\ &\quad\le\sum_{k=0}^{n}\left|\frac{\Gamma(\mu+\beta+1+k/\rho)}{\Gamma(\mu+k/\rho)}\right|\,|v(\zeta)|\,\frac{|z|^k}{|\zeta|^{k+1}}|\zeta|^{2k}\\ &\quad\le M\,\mathrm{const}(\beta,\rho,\varphi)\sum_{k=0}^{n}\frac{\Gamma(k+Re\beta+2)}{\Gamma(k+1)}\cdot\frac{|z|^k\,|\zeta|^k}{|\zeta|}\\ &\quad=\frac{\mathrm{const}}{|\zeta|}\sum_{k=0}^{n}\frac{\Gamma(k+Re\beta+2)}{\Gamma(k+1)}(|z|\,|\zeta|)^k\\ &\quad=\frac{\mathrm{const}}{|\zeta|}\cdot\frac{1}{(1-|z|\,|\zeta|)^{Re\beta+2}}\le\frac{\mathrm{const}}{|\zeta|(1-|z|)^{Re\beta+2}}\in L^1(\mathbb{D}).\end{aligned}$$

In the estimation above we use the following consequence of the Stirling's formula:

$$\frac{|\Gamma(\mu+R)|}{|\Gamma(\nu+R)|}\asymp R^{Re\mu-Re\nu},\quad R\to+\infty.$$

Thus

$$g(z)=-\frac{1}{\pi}\iint_{\mathbb{D}}\frac{v(\zeta)}{\zeta-z}dm(\zeta)+\sum_{k=0}^{\infty}c_kz^k\equiv g_1(z)+g_2(z),\quad z\in\mathbb{D}.\tag{17}$$

According to Theorem 1, we have $g_1\in C^k(\mathbb{D})$ and $\partial g_1(z)/\partial\bar{z}\equiv v(z)$, $z\in\mathbb{D}$. Further, since $g_2(z)$ is representable by a power series in $\mathbb{D}$, g_2 is holomorphic in $\mathbb{D}$, i.e. $\partial g_2(z)/\partial\bar{z}\equiv 0$, $z\in\mathbb{D}$. Hence, $g\in C^k(\mathbb{D})$ and g satisfies the Eq. (3). The proof is complete. □

Theorem 5. *Assume that $\alpha>-1$, $\gamma>-1$, $1\le p<+\infty$, $Re\beta\ge\alpha$ and $Re\varphi\ge\gamma$. Also let*

$$v(\zeta)\in C^k(\mathbb{D})\cap L^p_{\alpha+1,\rho,\gamma}(\mathbb{D})\tag{18}$$

for $k=1,2,3,\dots,\infty$. Then $g(z)\equiv g_{\beta,\rho,\varphi}(z)$ is of class $C^k(\mathbb{D})$ and satisfies the $\overline{\partial}$-equation (3).

Proof. First of all let's prove that under the assumptions of the theorem the integral (16) is convergent for every $z\in\mathbb{D}$. Close to the boundary, when $(1+|z|)/2\le|\zeta|<1$, we have (due to Proposition 3):

$$\begin{aligned}\frac{|v(\zeta)|}{|\zeta-z|}|Q(z;\zeta)|&\le\frac{2|v(\zeta)|}{1-|z|}\cdot\frac{\mathrm{const}(\beta,\rho,\varphi)(1-|\zeta|^{2\rho})^{Re\beta+1}}{(1-|z|)^{Re\beta+2}}\\ &\le\mathrm{const}(\beta,\rho,\varphi,z)|v(\zeta)|(1-|\zeta|^{2\rho})^{\alpha+1}.\end{aligned}\tag{19}$$

Hence, in view of (18) and the fact that $L^p_{\alpha,\rho,\gamma}(\mathbb{D})\subset L^1_{\alpha,\rho,\gamma}(\mathbb{D}), 1\le p<+\infty$, the convergence near the boundary was proved.

Now let's see the convergence in the neighborhood of z, when $z\neq 0$. Since $1/(\zeta-z)$ has integrable singularity and since v and Q are bounded near z then the convergence is obvious.

Finally, we have to show the convergence in the neighborhood of 0. We have two cases here.

Case 1. $z \neq 0$, then in view of Proposition 2 we have for $0 < |\zeta| < |z|/2$:

$$\frac{|v(\zeta)||Q(z;\zeta)|}{|\zeta - z|} \leq \frac{2\,M\,|Q(z;\zeta)|}{|z|} \leq \frac{2\,M\,\mathrm{const}(\beta,\rho,\varphi,z)}{|z|}\left(1+|\zeta|^{2Re\varphi+1}\right)$$
$$\leq \frac{2\,M\,\mathrm{const}(\beta,\rho,\varphi,z)}{|z|}\left(1+|\zeta|^{2\gamma+1}\right) \in L^1\left(0<|\zeta|<|z|/2\right),$$

where $M = \max\{|v(\zeta)| : |\zeta| \leq |z|/2\}$.

Case 2. $z = 0$, then in view of Proposition 2 we have for $0 < |\zeta| < 1/2$:

$$\frac{|v(\zeta)||Q(0;\zeta)|}{|\zeta|} \quad\leq\quad \frac{2\,M\,\mathrm{const}(\beta,\rho,\varphi)}{|\zeta|}\left(1+|\zeta|^{2\gamma+2}\right) \quad\in\quad L^1\left(0<|\zeta|<1/2\right).$$

Thus integral (16) is convergent, i.e. $g(z)$ is well-defined for every $z \in \mathbb{D}$. We have to show that $g \in C^k(\mathbb{D})$ and $\dfrac{\partial g(z)}{\partial \bar{z}} \equiv v(z)$, $z \in \mathbb{D}$. As these properties are local, it suffices to prove them in a neighborhood of an arbitrary point $z_0 \in \mathbb{D}$ (we intend to use the technique applied in the proof of Theorem 1.2.2 in [2]).

Let's take $0 < r_1 < r_2$ such that $D_1 = \{|\zeta - z_0| \leq r_1\} \subset D_2 = \{|\zeta - z_0| \leq r_2\} \subset \mathbb{D}$. In addition, if $z_0 \neq 0$, we assume that $r_1 < |z_0|/2$ and if $z_0 = 0$ we assume that $r_1 < 1/2$. Obviously, there exists a function $\psi \in C_c^\infty(\mathbb{D})$ such that

$$\psi|_{D_1} \equiv 1, \tag{20}$$

$$\psi|_{D\backslash D_2} \equiv 0, \tag{21}$$

$$\psi|_{D_2\backslash D_1} \in [0,1]. \tag{22}$$

Hence we can write:

$$g(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{v(\zeta)\;\psi(\zeta)}{\zeta - z} Q(z;\zeta)dm(\zeta)$$
$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{v(\zeta)\;(1-\psi(\zeta))}{\zeta - z} Q(z;\zeta)dm(\zeta) \equiv g_1(z) + g_2(z), \quad z \in \mathbb{D}. \tag{23}$$

From Theorem 4 we get that $g_1(z) \in C^k(\mathbb{D})$ (hence $g_1(z) \in C^k(\mathbb{D}_1)$) and

$$\frac{\partial g_1(z)}{\partial \bar{z}} \equiv v(z)\;\psi(z) \equiv v(z), \qquad z \in \mathbb{D}_1. \tag{24}$$

Since $\psi(\zeta) \equiv 1$, when $\zeta \in \mathbb{D}_1$, we can write

$$g_2(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}\backslash\mathbb{D}_1} \frac{v(\zeta)(1-\psi(\zeta))}{\zeta - z} Q(z;\zeta)dm(\zeta), \quad z \in \mathbb{D}. \tag{25}$$

We intend to show that $g_2 \in H(\mathbb{D}_1)$, i.e. g_2 is holomorphic in $\mathbb{D}_1$. To this end first let's note that for a fixed $\zeta \in \mathbb{D}\backslash\mathbb{D}_1$, the kernels $\dfrac{1}{\zeta - z}$ and $Q(z;\zeta)$ are holomorphic with respect to $z \in \mathbb{D}_1$. Consequently, it is sufficient to find a function $W(\zeta) \in L^1(\mathbb{D}\backslash\mathbb{D}_1)$ such that

$$\frac{|v(\zeta)|(1-\psi(\zeta))}{|\zeta - z|}|Q(z;\zeta)| \le W(\zeta) \tag{26}$$

uniformly with respect to $\zeta \in \mathbb{D}\backslash\mathbb{D}_1$ and z with $|z-z_0| \le r_0 < r_1$.

Obviously, $0 \le 1-\psi(\zeta) \le 1$, $\dfrac{1}{|\zeta - z|} \le \dfrac{1}{r_1 - r_0}$. Hence, we have to estimate $|v(\zeta)||Q(z;\zeta)|$.

Case 1. $z_0 \neq 0$. Put $\lambda = |z_0| + r_0 < 1$. Let's split $\mathbb{D}\backslash\mathbb{D}_1$ into 3 disjoint parts:

$$A = \left\{\zeta \in \mathbb{D}\backslash\mathbb{D}_1 : \frac{1+\lambda}{2} < |\zeta| < 1\right\},$$

$$B = \left\{\zeta \in \mathbb{D}\backslash\mathbb{D}_1 : |\zeta| < \frac{1}{2}\right\},$$

$$C = \left\{\zeta \in \mathbb{D}\backslash\mathbb{D}_1 : \frac{1}{2} \le |\zeta| \le \frac{1+\lambda}{2}\right\}.$$

If $|z-z_0| \le r_0$ and $\zeta \in A$, then $|z| \le |z_0| + r_0 = \lambda$. Hence $\dfrac{1+|z|}{2} < \dfrac{1+\lambda}{2}$. Then according to Proposition 3 we have

$$|Q(z;\zeta)| \le \frac{\text{const}(\beta,\rho,\varphi)}{(1-\lambda)^{Re\beta+2}}\left(1-|\zeta|^{2\rho}\right)^{\alpha+1}.$$

Hence

$$|v(\zeta)||Q(z;\zeta)| \le \text{const}(\beta,\rho,\varphi,\lambda)\ |v(\zeta)|\ \left(1-|\zeta|^{2\rho}\right)^{\alpha+1} \equiv W_1(\zeta).$$

In view of (18) we have $W_1 \in L^1(A)$.

If $|z-z_0| \le r_0$ and $\zeta \in B$, then according to Proposition 2

$$|Q(z;\zeta)| \le 1 + \frac{\text{const}(\beta,\rho,\varphi)}{(1-\lambda)^{Re\beta+1}}|\zeta|^{Re\varphi+1}$$

$$|v(\zeta)||Q(z;\zeta)| \le |v(\zeta)|(1+\text{const}(\beta,\rho,\varphi,\lambda)|\zeta|^{2\gamma+1}) \equiv W_2(\zeta)$$

and $W_2 \in L^1(B)$ as $v(\zeta)$ is bounded on $\left\{\zeta \in \mathbb{D}\backslash\mathbb{D}_1 : |\zeta| < \frac{1}{2}\right\}$.

If $|z-z_0| \le r_0$ and $\zeta \in C$, then note that

$$\left\{(z,\zeta) : |z-z_0| \le r_0 \quad \text{and} \quad \frac{1}{2} \le |\zeta| \le \frac{1+\lambda}{2}\right\}$$

is a compact set in the space $\mathbb{C}^2$. At the same time, as it follows from the proof of Proposition 1 of [13] the kernel $Q(z;\zeta)$ is a continuous function in variables $z \in \mathbb{D}$ and $\zeta \in \overline{\mathbb{D}}\backslash\{0\}$. Consequently, $|Q(z;\zeta)|$ is uniformly bounded in $\zeta \in C$ and z with $|z-z_0| \le r_0$. Also $|v(\zeta)|$ is uniformly bounded in $\zeta \in C$. Thus $|v(\zeta)||Q(z;\zeta)| \le M_1 \equiv W_3(\zeta) \in L^1(C)$. It remains to put

$$W(\zeta) = \begin{cases} W_1(\zeta), & \zeta \in A, \\ W_2(\zeta), & \zeta \in B, \\ W_3(\zeta), & \zeta \in C, \end{cases}$$

and note that W evidently belongs to $L^1(\mathbb{D}\backslash\mathbb{D}_1)$.

Case 2. $z_0 = 0$. Hence $|z| \leq r_0$. Similar to the previous case we write $\mathbb{D}\backslash\mathbb{D}_1$ as a union of the sets:

$$\widetilde{A} = \left\{ \zeta \in \mathbb{D} : \frac{1+r_0}{2} < |\zeta| < 1 \right\},$$

$$\widetilde{B} = \left\{ \zeta \in \mathbb{D} : r_1 \leq |\zeta| \leq \frac{1+r_0}{2} \right\}.$$

Then repeting the argument applied in Case 1, we get the necessary result. Thus $g_2(z) \in H(D_1)$, from which it follows that

$$\frac{\partial g_2(z)}{\partial \overline{z}} \equiv 0, \qquad z \in D_1.$$

This together with (24) implies that $g \in C^k(\mathbb{D}_1)$ and

$$\frac{\partial g(z)}{\partial \overline{z}} \equiv v(z), \quad z \in \mathbb{D}_1.$$

The proof is complete. □

Remark 1. *When $\beta = \alpha, \varphi = \gamma, p = 2$ and v satisfies (18) with α instead of $\alpha + 1$, Theorem 5 follows from [11], where the case of polydisc was considered.*

Received 09.03.2021
Reviewed 30.03.2021
Accepted 06.04.2021

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ՄԻԱՎՈՐ ՇՐՋԱՆՈՒՄ $\overline{\partial}$-ՀԱՎԱՍԱՐՄԱՆ ԿՇՌԱՅԻՆ ԼՈՒԾՈՒՄՆԵՐ

Հոդվածում դիտարկվում է $\partial g(z)/\partial\bar{z} = v(z)$ հավասարումը $\mathbb{D}$ միավոր շրջանում: C^k դասի $(k = 1,2,3,\ldots,\infty)$ այն v ֆունկցիաների համար, որոնք պատկանում են L^p–կշռային դասերին $(1 \le p < \infty)$ $|z|^{2\gamma}(1-|z|^{2\rho})^{\alpha}$, $z \in \mathbb{D}$, տիպի կշռային ֆունկցիայով, կառուցվում է լուծումների g_β ընտանիք (β–ն՝ կոմպլեքս պարամետր է):

Ф. В. АЙРАПЕТЯН

ВЕСОВЫЕ РЕШЕНИЯ $\overline{\partial}$-УРАВНЕНИЯ В ЕДИНИЧНОМ КРУГЕ

В статье рассматривается уравнение $\partial g(z)/\partial\bar{z} = v(z)$ в единичном круге $\mathbb{D}$. Для C^k-функций v $(k = 1,2,3,\ldots,\infty)$ из весовых L^p-классов $(1 \le p < \infty)$ с весовой функцией типа $|z|^{2\gamma}(1-|z|^{2\rho})^{\alpha}$, $z \in \mathbb{D}$, строится семейство решений g_β (β — комплексный параметр).