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M a t h e m a t i c s

ON NON-MONOTONOUS PROPERTIES OF SOME CLASSICAL AND NONCLASSICAL PROPOSITIONAL PROOF SYSTEMS

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We investigate the relations between the proof lines of non-minimal tautologies and its minimal tautologies for the Frege systems, the sequent systems with cut rule and the systems of natural deductions of classical and nonclassical logics. We show that for these systems there are sequences of tautologies ψ_n, every one of which has unique minimal tautologies φ_n such that for each n the minimal proof lines of φ_n are an order more than the minimal proof lines of ψ_n.

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Introduction. The minimal tautologies play a main role in the proof complexity area. Namely, the well-known propositional formulas (Pigeon Hole Principle, Clique Colouring pair, Topsy-Turvy Matrix, Hool's theorem, Ramsey theorem and some others), the proof complexities of which are investigated in many papers, are minimal tautologies. There is a traditional assumption that a non-minimal tautology shouldn't be less complicated than its minimal tautology, that means it must be some monotonicity of proofs. This idea was first revised by Anikeev in [1]. He introduced the notion of a monotonous proof system and gave examples of monotonous and non-monotonous systems, but both of them are not complete systems. In [2–5], the notion of strongly monotonous systems for propositional proof systems is additionally introduced and the properties of monotonous and strongly monotonous for many propositional proof systems of classical and nonclassical logics are investigated. Some of the investigated systems (resolution systems, cut-free sequent systems) are monotonous systems, in each of which the proof lines of non-minimal tautologies are not less than the proof lines of their minimal tautologies. Some others are not monotonous (systems based on the splitting method, elimination systems), the proof lines of some formulas can

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be less than the proof lines of some of their minimal tautologies. It was proved that many known systems are not strongly monotonous, but the question of monotonicity for some strong propositional systems still remains open. In [6] it is proved that the Frege systems of classical logic are not monotonous. In this paper we generalize this result for some other systems of classical logic (the sequent systems with cut rule and the systems of natural deductions), as well as for analogous systems of intuitionistic and Johansson's logics. It is shown that for each mentioned systems there are the sequences of tautologies ψ_n (respectively sequents $\to \psi_n$), each of which has unique minimal tautologies φ_n (respectively sequents $\to \varphi_n$) such that for every n the minimal proof lines of φ_n ($\to \varphi_n$) are an order more than the minimal proof lines of ψ_n ($\to \psi_n$).

Preliminaries. We will use the current concepts of propositional formula, subformula, elementary subformula, sequent, tautology in different logics, Frege proof systems, sequent systems, systems of natural deductions, proof and proof complexity [2–9]. Let us recall some of them.

Taking into account that in nonclassical logics none of logical connectives can be presented through the others, we assume that the language for the presented propositional formulas contains logical connectives $\neg$, $\&$, $\vee$, $\supset$ and may be constant $\bot$ (false) as well. Further we use also $A \simeq B$ for the presentation of formula $(A \supset B)\&(A \supset B)$.

The Complexity Properties of Formulas Proofs and Proofs Systems. We denote by $|\varphi|$ the size of the formula φ, defined as the number of all logical signs entries in it. It is obvious that the full size of the formula, which is understood to be the number of all symbols, is bounded by some linear function in $|\varphi|$.

Definition 1. *A tautology of some logic is called **minimal**, if the replacement result of all occurrences of each its non-elementary subformulas by some new variable is not a tautology of the same logic.*

Definition 2. *A minimal tautology φ of some logic is **minimal of some formula**, ψ if φ is ψ, or φ is the replacement result of all occurrences of some non-elementary subformulas of ψ by some new variable.*

We denote by $M(\psi)$ the set of all minimal tautologies of the tautology ψ.

In the theory of proof complexity one main characteristics of the proof is t**-complexity** (length), defined as the number of proof steps. We denote by $t^{\phi}(\varphi)$ the minimal possible value of t-complexity for all proofs of the tautology φ in the system ϕ .

Definition 3. *The proof system ϕ is called **t-monotonous**, if for every tautology ψ there is a minimal tautology φ, such that $\varphi \in M(\psi)$ and $t^{\phi}(\psi) = t^{\phi}(\varphi)$.*

Definition 4. *The proof system ϕ is called **t-strongly monotonous**, if for every tautology ψ there is no minimal tautology φ, such that $\varphi \in M(\psi)$ and $t^{\phi}(\varphi) > t^{\phi}(\psi)$.*

Definitions of Investigated Systems. Here we give the descriptions of the Frege systems, the sequent systems with cut rule and the systems of natural deductions for classical (**CL**), intuitionistic (**IL**) and Johansson's (**JL**) logics.

Definition 5. *A Frege system $\mathcal{FC}$ for **CL** uses a denumerable set of propositional variables, a finite, complete set of propositional connectives; $\mathcal{FC}$ has a finite set of inference rules defined by a figure of the form $\frac{A_1,A_2,\ldots,A_n}{B}$ (the rules of inference with zero hypotheses are the axioms schemes); $\mathcal{FC}$ must be sound and complete, i.e. for each rule of inference $\frac{A_1,A_2,\ldots,A_n}{B}$ every truth-value assignment, satisfying $A_1,A_2,\ldots,A_n$ also satisfies B, and $\mathcal{FC}$ must prove every tautology.*

Definition 6. *Two proof systems are called t-linearly equivalent, if any proof in one system can be modified to a proof of the same tautology in another system, so that the t-complexity of the proof is increased not more than linearly.*

In [8], it is proved that all Frege systems of **CL** are *t*-linearly equivalent.

For the definition of Frege systems for nonclassical logic, we must give some additional notions [9–13].

Definition 7. *The inference rule $\frac{A_1,A_2,\ldots,A_n}{B}$ is called **derivable** in the system ϕ, if the formula $(A_1 \supset (A_2 \ldots (A_n \supset B)\ldots))$ is provable in ϕ.*

Definition 8. *The inference rule $\frac{A_1,A_2,\ldots,A_n}{B}$ is called **admissible** in the system ϕ, if the formula B is provable in ϕ from premises $A_1,A_2,\ldots,A_n$.*

Note that for the systems of **CL** the derivability property of some rule implies the admissibility properties and vice versa, but for the nonclassical logics we have another situation : in [13] it is proved that there are some inference rules for the systems of **IL** (**JL**), which are admissible, but not derivable and it is proved in [11] ([12]) that the verification of the rule admissibility for the systems of **IL** (**JL**) can be done with the linear *t*- complexity in some specific system for **IL** (**JL**).

To give the definitions of Frege systems for **IL** (**JL**), we must fix the following main proof systems for classical, intuitionistic and Johansson's logics respectively by Cm, Im and Jm [9].

For each propositional formulas A,B,C the axiom schemas of the classical system Cm are:

1) $A \supset (B \supset A)$
2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3) $A \supset (B \supset A \wedge B))$
4) $A \wedge B \supset A; A \wedge B \supset B$
5) $A \supset A \vee B; B \supset A \vee B$
6) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
7) $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$
8) $\neg\neg A \supset A$

Inference rule is modus ponens $\dfrac{A, A \supset B}{B}$(m.p.).

In the system Im instead of axiom schema $\neg\neg A \supset A$ is taken $\neg A \supset (A \supset B)$, which is omited in the system Jm, where each formula $\neg A$ is replaced by $A \supset \bot$. Therefore, the axiom Schema 7 can be represented by the axiom Schema 2 and it must be omitted from Jm as well.

It is known that the *deduction theorem* holds in all three systems.

For each formula F we denote by J-image the formula, which is obtained from F by replacing each of its $\neg A$-type subformulas with formula $A \supset \bot$. Note, that the size of J-image of formula F can not be more than $2|F|$.

It is known that if some formula F is provable in Cm, then the formula $\neg\neg F$ is provable in Im, and if some formula F is provable in Im, then its J-image is provable in Jm.

Definition 9. *The Frege system $\mathcal{F}I$ ($\mathcal{F}J$) for **IL** (**JL**) is finite set of schematic axioms derivable in **IL** (**JL**) and schematic inference rules admissible in **IL** (**JL**) provided $\mathcal{F}I$ ($\mathcal{F}J$) contains up to a linear translation by lines to main system Im (Jm).*

Note that in [11, 12] the analogous definition is given with the last phrase "contains up to a polynomial translation by size to main system Im (Jm)" explaining the background for such definition, but it is the background for our Definition 9 as well.

As consequence we obtain

Proposition 1. *Any two Frege systems over $\neg, \&, \vee, \supset$ for **IL** are t-linearly equivalent [11]. Any two Frege systems over $\&, \vee, \supset, \bot$ for **JL** are t-linearly equivalent [12].*

Since the systems Cm, Im and Jm are Frege systems for **CL**, **IL** and **JL** respectively and it is proved in [9, 14] that the coresponding sequent systems with cut rule, the corresponding systems of natural deductions and mentioned main systems are t-linearly equivalent, from Proposition 1 it follows that

Proposition 2.

- *The Frege systems, the sequent systems with cut rule and the systems of natural deductions of **CL** are t-linearly equivalent.*
- *The Frege systems, the sequent systems with cut rule and the systems of natural deductions over $\neg, \&, \vee, \supset$ of **IL** are t-linearly equivalent.*
- *The Frege systems, the sequent systems with cut rule and the systems of natural deductions over $\&, \vee, \supset, \bot$ of **JL** are t-linearly equivalent.*

Important Formulas. In some papers on propositional proof complexity for 2-valued classical logic the following tautologies (Topsy-Turvy Matrix) play a key role

$$TTM_{n,m} = \bigvee_{(\sigma_1,\sigma_2,\ldots,\sigma_n)\in E^n} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}^{\sigma_j} (n \geq 1, 1 \leq m \leq 2^n - 1, E = 0, 1).$$

For all fixed $n \geq 1$ and m in the above indicated intervals every formula of this kind expresses the following true statement: given a 0,1-matrix of order $n \times m$ we can "topsy-turvy" some strings (writing 0 instead of 1 and 1 instead of 0) such that each column will contain at least one 1.

Let $A_n = TTM_{n,2^n-1}$. In [15], it is proved that in each Frege system of classical logic it holds the relation $t(A_n) = \Omega\left(\frac{|A_n|^{\frac{3}{2}}}{\log_2^3(|A_n|)}\right)$.

Note that the formula $\neg\neg A_n$ is provable in **IL**, its J-image is provable in **JL** and it is obvious that the minimal numbers of their proof lines in corresponding systems Im and Jm can not have less order than the proof lines of A_n in Cm.

Main Results. Here we prove that the mentioned systems of **CL**, **IL** and **JL** are non ***t*-monotonous**, consequently non ***t*-strongly monotonous**, but first we should give the following auxiliary statements relatively to the proof systems Cm, Im and Jm.

Lemma 1. *Let $P(A)$ be a tautology in Cm (Im, Jm), A be some of its subformulas of size n, and the variable q is not presented in $P(A)$. There is a modification of the subformula A to $A'(q)$, and there are not more than $2n$ tautologies $T_i(q)$, which can be proved in Cm (Im, Jm) in a constant number of steps, so that no minimal tautology of Cm (Im, Jm), can be obtained from the tautology $P'(A',q) = (q \supset P(A'(q)))\&T_1(q)\&\ldots\&T_i(q)\&\ldots$ by replacing with new variables neither any of the formulas $A'(q), T1(q), \ldots, T_i(q), \ldots$, nor their non-elementary subformulas.*

Proof. Let's describe the modification of each subformula U of A, while we will add with the conjunctions the formulas $T_i(q)$.

1. If U is a variable, then its modification coinsides with itself, that is $U' = U$. No new formula will be added.

2. If $U = (U_1 \supset U_2)$, and q is true in (1), then U is equivalent to $(U_1 \supset (q \supset U_2))$. The modification of U will be $U' = (U_1' \supset (q \supset U_2'))$, where U_1' and U_2' are modifications of U_1 and U_2, respectively. We will add with conjunctions the formulas $((\neg q) \supset (U_1' \supset (q \supset U_2')))$ and $((\neg q) \supset (q \supset U_2'))$ for the systems Cm, Im and its J-image for Jm. If we replace $(U'1 \supset (q \supset U'2))$ in P', then $((\neg q) \supset (U_1' \supset (q \supset U_2')))$ will not remain a tautology. If we replace $(q \supset U_2')$, then $((\neg q) \supset (q \supset U_2'))$ will not remain a tautology of corresponding system.

3. If $U = (\neg U_1)$, then we should modify U_1 to U_1'. The modified representation of U will be $U' = (\neg U_1')$. In case U_1' isn't a contradiction, we add with conjunctions the formula $(U_1' \supset (\neg(\neg U_1')))$ for the systems Cm, Im and its J-image for Jm, and otherwise the formula $((\neg(\neg U_1')) \supset U_1')$ for Cm, $(\neg(\neg((\neg(\neg U_1')) \supset U_1')))$ for Im and its J-image for Jm. If we replace $(\neg U_1')$ or $(\neg(\neg U_1'))$ in P', then the added formulas won't remain a tautology of the corresponding system.

4. If $U = (U_1 \& U_2)$, and q is true (1), then U is equivalent to $(U_1 \& (q \& U_2))$. The modification of U will be $U' = (U_1' \& (q \& U_2'))$, where U_1' and U_2' are the modifications of U_1 and U_2, respectively. Then we will add with the conjunctions the formulas $((U_1' \& (q \& U_2')) \supset q)$ and $((q \& U_2') \supset q)$ for the systems Cm, Im and its J-image for

Jm. If we replace $(U_1'\&(q\&U_2'))$ and (or) $(q\&U_2')$ in P', then at least one of the added formulas will not remain a tautology of the corresponding system.

5. If $U=(U_1\vee U_2)$, and q is true(1), then U is equivalent to $(U_1\vee((\neg q)\vee U_2))$. The modification of U will be $U'=(U_1'\vee((\neg q)\vee U_2'))$, where U_1' and U_2' are modifications of U_1 and U_2, correspondingly. We will add with conjunctions the following two formulas $((\neg q)\supset(U_1'\vee((\neg q)\vee U_2')))$ and $((\neg q)\supset((\neg q)\vee U_2'))$ for the systems Cm, Im and its J-image for Jm. If we replace $(U_1'\vee((\neg q)\vee U_2'))$ and (or) $((\neg q)\vee U_2')$ in P', then at least one of the added formulas will not remain a tautology of corresponding system.

Finally, we will add $q\supset$ to $P(A'(q))$, because we assumed q is true in (1) when modifying $\supset,\&,\vee$. □

Further, if the variable q is not represented in a formula A, then we call its modification $A'(q)$ for the systems Cm, Im and Jm by the method described in Lemma 1. q-discharged presentation of A.

Lemma 2. *Let A be a formula of size n for one of the systems Cm, Im or Jm and let $A'(q)$ be corresponding q-discharged representation. Then there exists a proof of $q\vdash A'(q)\supset A$ in $O(n)$ steps.*

Proof. For any formulas C and D each of the following formulas
$(q\supset((C\supset(q\supset D))\supset(C\supset D)))$, $(q\supset((C\supset D)\supset(C\supset(q\supset D))))$,
$(q\supset((C\&(q\&D))\supset(C\&D)))$, $(q\supset((C\&D)\supset(C\&(q\&D))))$,
$(q\supset((C\vee(\neg q\vee D))\supset(C\vee D)))$, $(q\supset((C\vee D)\supset(C\vee(\neg q\vee D))))$
is provable in Cm and Im in a constant number of steps. Therefore from premise q the following formulas
$((C\supset(q\supset D))\supset(C\supset D))$ and $((C\supset D)\supset(C\supset(q\supset D)))$
$((C\&(q\&D))\supset(C\&D))$ and $((C\&D)\supset(C\&(q\&D)))$
$((C\vee(\neg q\vee D))\supset(C\vee D))$ and $((C\vee D)\supset(C\vee(\neg q\vee D)))$
are provable in Cm and Im in a constant number of steps as well.

We have, that for any formulas F, F', E and E' the following formulas
$(F\simeq F')\supset((E\simeq E')\supset((F\supset E)\supset(F'\supset E')))$,
$(F\simeq F')\supset((E\simeq E')\supset((F\&E)\simeq(F'\&E')))$,
$(F\simeq F')\supset((E\simeq E')\supset(F\vee E)\simeq(F'\vee E'))$,
$(F\simeq F')\supset((\neg F)\simeq(\neg F'))$
also are provable in Cm and Im in a constant number of steps. Thus, constructing A step by step connecting to two equivalent formulas, not the old ones, but equivalent formulas (see Replacement Theorem 6 from [9]), we can prove in Cm and Im from premise q the formula $A'(q)\supset A$ with $O(n)$ steps. Analogous result for the system Jm we can obtain by repeating all the steps for the J-images of mentioned formulas. □

Lemma 3.

a) *If the tautology E is a minimal for the tautology C in the system Im (Jm), then tautology $\neg\neg E$ $(((E\supset\bot)\supset\bot))$ is a minimal for tautology $\neg\neg C(((C\supset\bot)\supset\bot))$;*

b) *for any tautology A of the system Im (Jm) the minimal steps of proof for the*

formula $\neg\neg A(((A \supset \bot) \supset \bot))$ in the system Im (Jm) can be more, than the minimal steps of proof for the formula A only with some constant.

Proof. Is obvious. □

Theorem 1. *The systems Cm, Im and Jm are not t-monotonous and, consequently, are not t-strongly monotonous.*

Proof. Consider the formula $P = A \vee (p \supset p)$, where p, q, t, r are not presented in A and $|A| = n$. Obviously, $t \vee (p \supset p)$ is minimal for P. If A is a minimal tautology, then $A \vee r$ is also minimal fo P. We can construct P' applying Lemma 12 to P:

$$P'_n(A', q) = (q \supset ((p \supset p) \vee A'(q)))\&T_1(q)\& \ldots \&T_i(q)\& \ldots$$

It's obvious that $P'_n(A', q)$ can be proved in $O(n)$ steps and its unique minimal is the following:

$$B_n(q, r, A') = (q \supset (r \vee A'(q)))\&T_1(q)\& \ldots \&T_i(q)\& \ldots$$

Suppose it can be proved in Cm in $f(n)$ steps. Then the following three tautologies $B_n(q, \neg r, A')$, $B_n(\neg q, r, A')$, $B_n(\neg q, \neg r, A')$ can be proved in at most $f(n)$ steps as well.

Now we can prove that any minimal tautology A is proved in Cm with $O(f(n) + |A|)$ steps.

Let the formula $B_n(q, r, A')$ is proved already. Then in $O(n)$ steps by &-elimination rule we can prove the formula $(q \supset (r \vee A'(q)))$. Now we add two premises $q, \neg r$ and continue the Proof:

$$\begin{array}{rll}
q, \neg r & \vdash & (q \supset (r \vee A'(q))) \\
 & \vdash & q \\
 & \vdash & r \vee A'(q) \\
 & \vdash & \neg r \\
 & \vdash & \neg r \supset ((r \vee A'(q)) \supset A'(q)) \quad \text{tautology(*)} \\
 & \vdash & ((r \vee A'(q)) \supset A'(q)) \\
 & \vdash & A'(q) \\
\ldots\ldots\ldots\ldots & & \\
 & \vdash & A'(q) \supset A \quad \text{by Lemma 2} \\
 & \vdash & A \\
q & \vdash & \neg r \supset A
\end{array}$$

By analogy, after proving the formulas $B_n(q, \neg r, A')$, $B_n(\neg q, r, A')$, $B_n(\neg q, \neg r, A')$, we obtain: $q, \neg\neg r \vdash (q \supset (\neg r \vee A'(q))) \ldots q \vdash \neg\neg r \supset A$ consequently, using the proof of tautology $(\neg r \supset A) \supset ((\neg\neg r \supset A) \supset A)$ (**), we obtain $q \vdash A$.

Further by analogy $\neg q, \neg r \vdash (\neg q \supset (r \vee A'(\neg q))) \ldots \neg q \vdash \neg r \supset A$ and $\neg q, \neg\neg r \vdash (\neg q \supset (\neg r \vee A'(\neg q))) \ldots \neg q \vdash \neg\neg r \supset A \ldots$, $\neg q \vdash A$ and finally, using the proof of tautology $(q \supset A) \supset ((\neg q \supset A) \supset A)$ (***) we prove A.

Thus, we got a proof of any minimal tautology of size n with $O(f(n) + n)$ steps. On the other hand, as it was noted in "'Important Formulas', there exists a sequence of minimal tautologies A_n, such that in each Frege system of classical logic

$t(A_n) = \Omega\left(\frac{|A_n|^{\frac{3}{2}}}{\log_2^3(|A_n|)}\right)$ and consequently, $f(n) = \Omega\left(\frac{n^{\frac{3}{2}}}{\log_2^3(n)}\right)$. So, for the system Cm we constructed the tautologies $P'_n(A',q)$, which are proved in $O(n)$ steps, while its unique minimal tautologies $B_n(q,r,A')$ in $\Omega\left(\frac{n^{\frac{3}{2}}}{\log_2^3(n)}\right)$ steps.

Note that we use the classical tautology (*) and results of the substitutions in it, which are tautology in Im and its J-images are tautologies in Jm, but the classical tautologies (**) and (***) are not provable in Im. Therefore, to prove the theorem for the system Im and Jm, we need to use the tautologies $(\neg r \supset A) \supset ((\neg\neg r \supset A) \supset \neg\neg A)$, $(q \supset A) \supset ((\neg q \supset A) \supset \neg\neg A)$ for Im and its J-images for Jm, which give us the proof of tautology $\neg\neg A$ for any minimal tautology A for Im and J-image of A for Jm, as well as Lemma 3. □

Theorem 2.

a) *The Frege systems, the sequent systems with cut rule and the systems of natural deductions of* ***CL*** *are not t-monotonous, and consequently, are not t-strongly monotonous.*

b) *The Frege systems, the sequent systems with cut rule and the systems of natural deductions over* $\neg, \&, \vee, \supset$ *of* ***IL*** *are not t-monotonous and, consequently, are not t-strongly monotonous.*

c) *The Frege systems, the sequent systems with cut rule and the systems of natural deductions over* $\&, \vee, \supset, \bot$ *of* ***JL*** *are not t-monotonous and, consequently, are not t-strongly monotonous.*

Proof. Follows from Theorem 1 and Proposition 2. Note, that for the sequent systems with cut rule and the systems of natural deductions for any minimal tautology A it is proved the corespondent sequents. □

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ԱՍՈՒՅԹԱՅԻՆ ՀԱՇՎԻ ԴԱՍԱԿԱՆ ԵՎ ՈՉ ԴԱՍԱԿԱՆ ՈՐՈՇ ԱՐՏԱԾՄԱՆ ՀԱՄԱԿԱՐԳԵՐԻ ՈՉ ՄՈՆՈՏՈՆՈՒԹՅԱՆ ՎԵՐԱԲԵՐՅԱԼ

Սույն աշխատանքում հետազոտվել են ոչ մինիմալ նույնաբանությունների և նրանց մինիմալների նվազագույն արտածման քայլերի հարաբերությունը դասական և ոչ դասական տրամաբանությունների Ֆրեգեի համակարգերում, հափույթի կանոնով սեկվենցիալ համակարգերում և բնական արտածումների

համակարգերում: Ապացուցված է, որ դիտարկված համակարգերից յուրաքանչյուրում գոյություն ունի այնպիսի ψ_n ոչ մինիմալ նույնաբանությունների հաջորդականություն, որոնցից յուրաքանչյուրը ունի միակ այնպիսի մինիմալ φ_n նույնաբանություն, որի արտածումների նվազագույն քայլերի քանակը ըստ կարգի ավելի մեծ է ψ_n բանաձևերի արտածումների նվազագույն քայլերի քանակից:

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О СВОЙСТВЕ НЕМОНОТОННОСТИ НЕКОТОРЫХ КЛАССИЧЕСКИХ И НЕКЛАССИЧЕСКИХ ПРОПОЗИЦИОНАЛЬНЫХ СИСТЕМ ВЫВОДОВ

В настоящей работе для систем Фреге, секвенциальных систем с правилом сечения и систем натуральных выводов классической и неклассических логик исследовано соотношение количества шагов выводов неминимальных тавтологий и их минимальных тавтологий. Доказано, что для каждой из рассмотренных систем существуют такие последовательности неминимальных тавтологий ψ_n, каждая из которых имеет единственную минимальную φ_n и для каждого n наименьшее количество шагов выводов фомул φ_n по порядку больше наименьшего количества шагов выводов фомул ψ_n.