PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2022, **56**(1), p. 1–6

Mathematics

ON SCHUR MULTIPLIER OF SOME RELATIVELY FREE GROUPS

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In this paper central extensions of free groups of infinitely based varieties of S.I. Adian are constructed. Using this extensions we prove that the Schur multipliers of mentioned relatively free groups are free Abelian groups of infinite rank. It is well-known that these varieties are given by identities in two variables. For a fixed rank m, the set of free groups of rank m of these varieties has the cardinality of continuum.

https://doi.org/10.46991/PYSU:A/2022.56.1.001

MSC2010: Primary: 20E22; Secondary: 20F05.

Keywords: central extension, infinitely based variety, Schur multiplier, relatively free group.

Introduction. The aim of the work is to construct and study central extensions of free groups of infinitely based varieties of S.I. Adian by free Abelian groups. It is known that these varieties are given by the following system of identities in two variables

$$\{(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1\}, \tag{1}$$

where the parameter p runs through all prime numbers and $n \geq 1003$ is an arbitrary fixed odd number. Eq. (1) system is independent, that is, none of these identities is a consequence of the others (the question of the existence of such systems was posed by B. Neumann in 1937). This implies that for any odd $n \geq 1003$ there exists a continuum of different varieties $\mathcal{A}_n(\Pi)$ corresponding to different sets of primes Π, if we require that $p \in \Pi$. At the same time, for fixed $m > 1$ there exists a continuum of non-isomorphic groups $\Gamma(m,n,\Pi)$, where $\Gamma(m,n,\Pi)$ is a relatively free group of rank m of the variety $\mathcal{A}_n(\Pi)$. These varieties were first constructed by S.I. Adian in [1, 2]. Their detailed description is also contained in the monograph [3]. A number of new interesting properties of free groups $\Gamma(m,n,\Pi)$ of $\mathcal{A}_n(\Pi)$ varieties were obtained recently in [4, 5].

A.Yu. Olshansky and A. Ashamanov proved in [6] (see also [7]) that the Schur multiplier of a free Burnside group $B(m,n)$ of finite rank $m > 1$ of odd period $n > 10^{10}$

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is a free Abelian group of infinite rank. Recall, that if the group $G = F/R$ is presented in terms of a free group F on a set of generators, and a normal subgroup R generated by a set of relations on the generators, then the group

$$M(G) = R \cap [F,F]/[R,F]$$

is called the Schur multiplier of G. As shown in [8], this result can be extended to all odd periods $n \geq 1003$. The proof is based on the study of central extensions of Burnside groups. Here we show that the approach of [6, 8] can be generalized and extended to the groups $\Gamma(m,n,\Pi)$.

An inductive definition of the groups $\Gamma(m,n,\Pi)$ is given in [3] (Chap. VII). We will not repeat this definition (due to its cumbersomeness), assuming that the reader is familiar with its definition. In future, without special references, we will use the notation and terminology of the monograph [3] and the articles [4, 9].

Formulation of Results. Fix the finite alphabet $X = a_1, ..., a_m, a_1-1, ..., a_m-1$, $m > 1$, and consider in this alphabet a set of elementary words as defined in [3] (§2 of Chap. VII). Each elementary period is either marked in some rank or unmarked. In [4] it is proved that each marked elementary period has order n in $G = \Gamma(m,n,\Pi)$, while unmarked elementary period has infinite order (see [4], Lemmas 4, 5).

Denote by $\mathcal{E}_\alpha$ the set, consisting of all marked elementary periods of rank α, and by $\mathcal{E}$ we denote the set of all marked elementary periods:

$$\mathcal{E} = \bigcup_{\alpha=1}^{\infty} \mathcal{E}_\alpha. \tag{2}$$

Since the set of identities of the group $G = \Gamma(m,n,\Pi)$ is countable, G cannot be defined by a finite number of defining relations.Therefore, the set of $\mathcal{E}$ is also countable. Fix some numbering and let $\mathcal{E} = \{A_j | j \in \mathbb{N}\}$ (where $\mathbb{N}$ is the set of natural numbers).

Fix also a free Abelian group $\mathcal{D}$ given by generating and defining relations:

$$\mathcal{D} = \langle d_1, d_2, \ldots, d_i, \ldots \mid \forall i, j\ [d_i, d_j] = 1 \rangle. \tag{3}$$

Denote by $A_{\mathcal{D}}(G)$ the group given by the system of generators of two types

$$a_1, a_2, \ldots, a_m \tag{4}$$

and

$$d_1, d_2, \ldots., d_i, \ldots, \tag{5}$$

and a system of the following defining relations:

$$\forall i, j\ [d_i, d_j] = 1, \tag{6}$$

$$a_i d_j = d_j a_i, \tag{7}$$

$$A_j^n = d_j \tag{8}$$

for all $i = 1, 2, \ldots, m$, $j \in \mathbb{N}$ and $A_j \in \mathcal{E}$.

It follows from the relations Eq. (8) that the group $A_{\mathcal{D}}(G)$ is generated by generators Eq. (4).

Theorem 1. *For any $m > 1$ and odd $n \geq 1003$:*

1) *the center of the group $A_{\mathcal{D}}(G)$ coincides with $\mathcal{D}$;*

2) *the quotient group of the group $A_{\mathcal{D}}(G)$ by the subgroup $\mathcal{D}$ is the group $G = \Gamma(m,n,\Pi)$;*

3) *verbal subgroup of the group $A_{\mathcal{D}}(G)$ corresponding to the word system $\{[x^{pn},y^{pn}]^n\}$, $p \in \Pi$, coincides with the Abelian group $\mathcal{D}$ Eq. (3).*

Theorem 2. *The group $A_{\mathcal{D}}(G)$ is a free group of rank m in the variety of groups $\mathfrak{D}$ defined by the system of identities $[[x^{pn},y^{pn}]^n,z] = 1$, $p \in \Pi$.*

Thus, by Theorem 2, $A_{\mathcal{D}}(G) = F_m/[F_m,N]$, where F_m is a free group of rank m and N is its verbal subgroup generated by the system of words $[[x^{pn},y^{pn}]^n,z]$, $p \in \Pi$. From point 3) of Theorem 1 it immediately follows

Corollary 1. *The center of the group $A_{\mathcal{D}}(G)$ is a free Abelian group of countable rank. Moreover, the elements $\{d_j | j \in \mathbb{N}\}$ are free generators of the center $A_{\mathcal{D}}(G)$.*

Theorems 1, 2 also imply

Corollary 2. *The Schur multiplier of the free group of the variety generated by the system of identities $\{[x^{pn},y^{pn}]^n = 1\}$, $p \in \Pi$, is a free Abelian group of countable rank for all finite ranks $m > 1$ and odd periods $n \geq 1003$.*

Proof. By definition, the Schur multiplier M in the variety of groups generated by the system of identities $[x^{pn},y^{pn}]^n = 1$, $p \in \Pi$, is the quotient group

$$M = N \cap [F_m,F_m]/[F_m,N],$$

where F_m is a free group of rank m and N is its verbal subgroup generated by the word system $[x^{pn},y^{pn}]^n$, $p \in \Pi$. Obviously, $M < N/[F_m,N]$. By conditions 2) and 3) of Theorem 1, the group $N/[F_m,N]$ coincides with the center of the group $A_{\mathcal{D}}(G)$. By Corollary 1, $N/[F_m,N]$ is a free Abelian group of countable rank.

The quotient group of the group $N/[F_m,N]$ by the subgroup M is isomorphic to the group

$$N/N \cap [F_m,F_m] \simeq N[F_m,F_m]/[F_m,F_m],$$

which is a subgroup of free Abelian group $F_m/[F_m,F_m]$ of rank m. Hence, the Schur multiplier M is also a free Abelian group of countable rank. □

The proof of Corollary 2 essentially repeats the elegant proof of the analogous assertion for free Burnside groups from [6]. We have included it here for shortness of it's proof.

Proofs of Theorem 1.

Lemma 1. *Subgroup $\mathcal{Z}$ generated by the elements $\{d_j | j \in \mathbb{N}\}$, coincides with the center of the group $A_{\mathcal{D}}(G)$, and the quotient $A_{\mathcal{D}}(G)/\mathcal{Z}$ is isomorphic to the group G.*

Proof. From the relations (6) and (7) it follows that $\mathcal{Z}$ is contained in the center of the group $A_{\mathcal{D}}(G)$ and the quotient group of the group $A_{\mathcal{D}}(G)$ by the subgroup

$\mathcal{Z}$ is the group G. By Theorem 1 [4], the centralizer of any nontrivial element of the relatively free group G is a cyclic group. Thus, the center of the group G is trivial. Therefore, the quotient group $A_{\mathcal{D}}(G)/\mathcal{Z}$ has a trivial center and $\mathcal{Z}$ coincides with the center of the group $A_{\mathcal{D}}(G)$. □

Lemma 2. *The group $A_{\mathcal{D}}(G)$ satisfies the system of identities $[[x^{pn},y^{pn}]^n,z]=1$, $p\in\Pi$, and the verbal subgroup of the group $A_{\mathcal{D}}(G)$ corresponding to the word system $[x^{pn},y^{pn}]^n$, $p\in\Pi$, coincides with $\mathcal{Z}$ in the group $A_{\mathcal{D}}(G)$.*

Proof. From the Lemma 1 it follows that in $A_{\mathcal{D}}(G)$ the identity system of

$$[[x^{pn},y^{pn}]n,z]=1,\ p\in\Pi.$$

Moreover, since the center $\mathcal{Z}$ is generated by the elements $\{d_j | j\in\mathbb{N}\}$, then from the relations (8) and from the definition of marked elementary periods (see [3], Section 2.2, Chap. VII) it follows that the verbal subgroup of the group $A_{\mathcal{D}}(G)$ generated by the system of words $[x^{pn},y^{pn}]^n$, $p\in\Pi$, is the subgroup $\mathcal{Z}$. □

Eliminating all letters d_j for $j\in\mathbb{N}$ from the system of generators of the group $A_{\mathcal{D}}(G)$, we have for the groups $A_{\mathcal{D}}(G)$ get the following presentation:

$$A_{\mathcal{D}}(G)=\langle a_1,a_2,\dots,a_m \,|\, [A_j^n,a_k]=1,\ j\in\mathbb{N},\ k=1,2,\dots,m\rangle.$$

This implies that each relation of the group $A_{\mathcal{D}}(G)$ is a consequence of the system of identities $[[x^{pn},y^{pn}]^n,z]=1$, $p\in\Pi$. On the other hand, by Lemma 2, $A_{\mathcal{D}}(G)$ satisfies the identities $[[x^{pn},y^{pn}]^n,z]=1$, $p\in\Pi$. Thus the group $A_{\mathcal{D}}(G)$ is the free group of rank m of the variety $\mathfrak{D}$ defined by the system of identities $[[x^{pn},y^{pn}]^n,z]=1$, $p\in\Pi$.

Theorem 1 is proved.

Proofs of Theorem 2. According to Lemmas 1 and 2, the subgroup $\mathcal{Z}$ generated by the elements $\{d_j | j\in\mathbb{N}\}$ coincides with the center of the group $A_{\mathcal{D}}(G)$, the quotient group $A_{\mathcal{D}}(G)/\mathcal{Z}$ is isomorphic to the group $G=\Gamma(m,n,\Pi)$, the verbal subgroup of the group $A_{\mathcal{D}}(G)$ corresponding to the word system $[x^{pn},y^{pn}]^n$, $p\in\Pi$, coincides with $\mathcal{Z}$, and the group $A_{\mathcal{D}}(G)$ satisfies the system of identities $[[x^{pn},y^{pn}]^n,z]=1, p\in\Pi$. Therefore, Theorem 1 will be proved if we show that the Abelian group $\mathcal{D}$ with the same generators $\{d_j | j\in\mathbb{N}\}$ is embedded in the group $A_{\mathcal{D}}(G)$ and, thus, coincides with $\mathcal{Z}$. To prove the last assertion, we only need to literally repeat § 3–5 of the paper [8]. In this case, it is necessary everywhere to mean by an elementary period of rank α a marked elementary period of the same rank. After such an agreement, the assertions of § 3–5 of [8] can also be applied to the groups we are considering.

Theorem 2 is proved.

Received 07.02.2022
Reviewed 24.02.2022
Accepted 07.03.2022

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Գ. Գ. ԳԵՎՈՐԳՅԱՆ

ՈՐՈՇ ՀԱՐԱԲԵՐԱԿԱՆ ԱԶԱՏ ԽՄԲԵՐԻ ՇՈՒՐԻ ԲԱԶՄԱՊԱՏԿԻՉՆԵՐԻ ՄԱՍԻՆ

Աշխատանքում կառուցվում են անվերջ բազիսով Ս.Ի. Ադյանի բազմաձևությունների ազատ խմբերի կենտրոնական ընդլայնումները: Կիրառելով այդ ընդլայնումները, ապացուցում ենք, որ նշված հարաբերական ազատ խմբերի Շուրի բազմապատկիչները անվերջ ռանգի ազատ Աբելյան խմբեր են: Հայտնի է, որ այդ բազմաձևությունները տրվում են երկու փոփոխականի նույնություններով: Սևեռված m-ի դեպքում, այդ բազմաձևությունների m ռանգի ազատ խմբերի բազմության հզորությունը կոնտինիում է:

Г. Г. ГЕВОРКЯН

О МУЛЬТИПЛИКАТОРАХ ШУРА НЕКОТОРЫХ ОТНОСИТЕЛЬНО СВОБОДНЫХ ГРУПП

В работе строятся центральные расширения свободных групп бесконечно базирующих многообразий С.И. Адяна. С помощью этих расширений доказывается, что мультипликаторы Шура указанных относительно свободных групп являются свободными абелевыми группами бесконечного ранга. Хорошо известно, что эти многообразия задаются тождествами от двух переменных. Для фиксированного ранга m мощность множества свободных групп ранга m этих многообразий есть континуум.