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Mathematics

ON AUTOMORPHISM GROUPS OF ENDOMORPHISM SEMIGROUPS OF FINITE ELEMENTARY ABELIAN GROUPS

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In this article, we explore the automorphisms of endomorphism semigroups and automorphism groups of the finite elementary Abelian groups. In particular, we prove that $\mathrm{Aut}(\mathrm{End}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p))$ can be canonically embedded into $\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p))$ using an elementary approach based on matrix operations. We also show that all automorphisms of $\mathrm{End}(\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p)$ are inner.

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Introduction. For a group G, let $\mathrm{Aut}(G)$ and $\mathrm{End}(G)$ denote the automorphism group and endomorphism semigroup of G, respectively.

The question about description of the automorphisms of $\mathrm{End}(A)$, for A being a free algebra in a certain variety, was raised by B.I. Plotkin (see [1]).

The automorphisms of $\mathrm{End}(G)$ and the relationship between $\mathrm{Aut}(\mathrm{End}(G))$ and $\mathrm{Aut}(\mathrm{Aut}(G))$ are well-studied for some specific groups [2–4].

In this paper, we study the automorphism group of endomorphism semigroup of finite elementary Abelian groups.

Let $G = \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{n}$, where p is a prime number and $n \geq 2$.

In contrast to the results for the free groups [2] and free Burnside groups [3], we show that $\mathrm{Aut}(\mathrm{End}(G))$ and $\mathrm{Aut}(\mathrm{Aut}(G))$ are not isomorphic in general. More precisely, the former one can be canonically embedded into the latter.

It can be easily checked that $\mathrm{End(G)} \cong M_n(\mathbb{Z}_p)$ and $\mathrm{Aut(G)} \cong GL_n(\mathbb{Z}_p)$, where $M_n(\mathbb{Z}_p)$ denotes the multiplicative semigroup of all $n \times n$ matrices over $\mathbb{Z}_p$.

It should be noted that the automorphisms of $M_n(\mathbb{Z}_p)$ have been already described in [5,6]. Here, we introduce a new approach for this problem, which was motivated by [3, 7]. The method is quite elementary: it uses only operations with matrices.

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Now we state our main results.

Consider the natural homomorphism

$$\tau : \mathrm{Aut}(\mathrm{End}(G)) \to \mathrm{Aut}(\mathrm{Aut}(G)),$$

given by $\tau(\varphi) = \varphi\big|_{\mathrm{Aut}(G)}$ for any automorphism φ of $\mathrm{End}(G)$.

Theorem 1. *τ is an injective homomorphism.*

Theorem 2. *All automorphisms of $M_n(\mathbb{Z}_p)$ are inner.*

Preliminaries. We start with introducing some notation to be used throughout the paper.

Let E_{ij} denote the $n \times n$ matrix with only nonzero element 1 in the (i, j)-th position, and set $N = E_{12} + E_{23} + \cdots + E_{n-1n} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$

Let P_{ij} denote the permutation matrix corresponding to the transposition $(i\ j)$.

For a matrix M, $[M]_{ij}$ stands for the entry in the i-th row and j-th column.

Lemma 1. *$E_{ij}E_{kl} = \delta_{jk}E_{il}$, where $\delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$.*

The proof of Lemma 1 is straightforward.

Lemma 2. *If $\varphi \in \mathrm{Aut}(M_n(\mathbb{Z}_p))$, then $\varphi(O_n) = O_n$, where O_n denotes the $n \times n$ zero matrix.*

Proof. Indeed, $O_n\varphi^{-1}(O_n) = O_n$ implies $\varphi(O_n)O_n = \varphi(O_n)$, therefore, $\varphi(O_n) = O_n$. □

Next lemma is crucial for the method of proof of the main result.

Lemma 3. *$M_n(\mathbb{Z}_p)$ can be generated by 3 elements, one of which is N, and the other two are invertible.*

Proof. See [7] for the proof. □

The Proof of the Main Results.

Proof of Theorem 1. Since End(G) $\cong M_n(\mathbb{Z}_p)$ and Aut(G) $\cong GL_n(\mathbb{Z}_p)$, we are going to prove that $\ker(\tau) = \{\mathbb{1}_{M_n(\mathbb{Z}_p)}\}$, that is, if the restriction of $\varphi : M_n(\mathbb{Z}_p) \to M_n(\mathbb{Z}_p)$ on $GL_n(\mathbb{Z}_p)$ is the identity automorphism of $GL_n(\mathbb{Z}_p)$, then φ is the identity automorphism of $M_n(\mathbb{Z}_p)$.

So, assume $\varphi\big|_{\mathrm{Aut}(G)} = \mathbb{1}_{GL_n(\mathbb{Z}_p)}$.

Considering Lemma 3, it is sufficient to show that $\varphi(N) = N$.

Let $\varphi(N) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$

We consider the $n \times n$ permutation matrix corresponding to the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}: \quad P_1 = E_{1n} + E_{21} + \dots + E_{nn-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Since $NE_{11} = (E_{12} + \dots + E_{n-1n})E_{11} = O_n$ by Lemma 1, we have

$$\begin{aligned} NP_1 = N(P_1 + E_{11}) = (E_{12} + \dots + E_{n-1n})(E_{1n} + E_{21} + \dots + E_{nn-1}) = \\ = E_{11} + E_{22} + \dots + E_{n-1n-1} =: A, \end{aligned} \tag{1}$$

and hence $\varphi(NP_1) = \varphi(N(P_1 + E_{11})) = \varphi(A)$. On the other hand,

$$\varphi(A) = \varphi(N)\varphi(P_1) = \varphi(N)P_1 = \begin{pmatrix} a_{12} & \cdots & a_{1n} & a_{11} \\ a_{22} & \cdots & a_{2n} & a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} & a_{n1} \end{pmatrix}, \tag{2}$$

because $P_1 \in GL_n(\mathbb{Z}_p)$.

$$\begin{aligned} \varphi(N(P_1 + E_{11})) = \varphi(N)\varphi(P_1 + E_{11}) = \varphi(N)(P_1 + E_{11}) = \\ = \begin{pmatrix} a_{11} + a_{12} & \cdots & a_{1n} & a_{11} \\ a_{21} + a_{22} & \cdots & a_{2n} & a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + a_{n2} & \cdots & a_{nn} & a_{n1} \end{pmatrix}, \end{aligned} \tag{3}$$

since $P_1 + E_{11}$ is also invertible.

Combining (1), (2) and (3), we obtain the following equations:

$$a_{i1} + a_{i2} = a_{i2}, \quad i = 1, 2, \dots, n,$$

hence $a_{11} = a_{21} = \dots = a_{n1} = 0$.

By a similar reasoning, we have

$$P_1N = (P_1 + E_{nn})N = E_{22} + \dots + E_{nn} =: B. \tag{4}$$

So $\varphi(P_1N) = \varphi((P_1 + E_{nn})N) = \varphi(B)$. But

$$\varphi(P_1N) = \varphi(P_1)\varphi(N) = P_1\varphi(N) = \begin{pmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n} \end{pmatrix} \tag{5}$$

and

$$\begin{aligned} \varphi((P_1 + E_{nn})N) = \varphi(P_1 + E_{nn})\varphi(N) = (P_1 + E_{nn})\varphi(N) = \\ = \begin{pmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-11} + a_{n1} & a_{n-12} + a_{n2} & \cdots & a_{n-1n} + a_{nn} \end{pmatrix}, \end{aligned} \tag{6}$$

since $P_1 + E_{nn} \in GL_n(\mathbb{Z}_p)$.

Eqs. (4), (5) and (6) together imply the following equations:

$$a_{n-1j} + a_{nj} = a_{n-1j}, \quad j = 1, 2, \ldots, n,$$

hence $a_{n1} = a_{n2} = \cdots = a_{nn} = 0$.

Thus, we have

$$\varphi(N) = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n-12} & \cdots & a_{n-1n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \varphi(A) = \begin{pmatrix} a_{12} & \cdots & a_{1n} & 0 \\ a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-12} & \cdots & a_{n-1n} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We first consider the case $n = 2$. Then we have

$$\varphi(A) = \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a_{12} & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 2 asserts that a_{12} is nonzero.

As $A^{p-1} = A$, we get $\varphi(A) = \begin{pmatrix} a_{12}^{p-1} & 0 \\ 0 & 0 \end{pmatrix} = A$ by Fermat's little theorem. Therefore, $\varphi(N) = N$ and we are done.

Below we assume that $n > 2$. It is easy to check that $P_{1n}BP_{1n} = A$. Since $B = P_1N$, we will have

$$(P_{1n}P_1)NP_{1n} = A. \tag{7}$$

$$P_{1n}P_1 = (E_{1n} + E_{22} + \cdots + E_{n-1n-1} + E_{n1})(E_{1n} + E_{21} + \cdots + E_{nn-1}) =$$
$$= E_{1n-1} + E_{21} + \cdots + E_{n-1n-2} + E_{nn}.$$

Applying φ to (7), we get the following equality: $P_2P_1\varphi(N)P_2 = \varphi(A)$.

Writing $\varphi(N)$ in terms of E_{ij}, we easily obtain

$$(P_{1n}P_1)\varphi(N)P_{1n} =$$
$$(E_{1n-1}+E_{21}+\cdots+E_{n-1n-2}+E_{nn})\left(\sum_{i,j=1}^{n} a_{ij}E_{ij}\right)(E_{1n}+E_{22}+\cdots+E_{n-1n-1}+E_{n1}) =$$
$$(a_{n-1n}E_{11} + a_{n-12}E_{12} + \cdots + a_{n-1n-1}E_{1n-1}) + (a_{1n}E_{21} + a_{2n}E_{31} + \cdots + a_{n-2n}E_{n-1}) +$$
$$+ \sum_{i=1}^{n-2}\sum_{j=2}^{n-1} a_{ij}E_{i+1j}.$$

On the other hand, $\varphi(A) = \sum_{i=1}^{n}\sum_{j=2}^{n} a_{ij}E_{ij+1}$ by (2), consequently, we get

$$\begin{cases} a_{12} = a_{23} = \ldots = a_{n-1n}, \\ a_{13} = a_{24} = \ldots = a_{n-12}, \\ \cdots \\ a_{1n} = a_{22} = \ldots = a_{n-1n-1}. \end{cases}$$

Thus we have $\varphi(A) = \begin{pmatrix} a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ a_{1n} & a_{12} & \cdots & a_{1n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{13} & a_{14} & \cdots & a_{12} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$.

It is obvious that $P_{2k}AP_{2k} = A$ and therefore, $P_{2k}\varphi(A)P_{2k} = \varphi(A)$. But since $[P_{2k}\varphi(A)P_{2k}]_{12} = a_{1k+1}$, it follows that $a_{13} = \cdots = a_{1n}$.

Now let $a = a_{12}, b = a_{13}$, so

$$\varphi(A) = \begin{pmatrix} a & b & \cdots & b & 0 \\ b & a & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(N) = \begin{pmatrix} 0 & a & b & \cdots & b \\ 0 & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b & b & \cdots & a \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We show that $a = 1$ and $b = 0$.

For $p = 2$, we consider the following possible cases:

1. $a = b = 0$. Since φ is an automorphism, $\varphi(A) = \varphi(N) = O_n$ yields a contradiction.
2. $a = b = 1$. If n is even, $(\varphi(N))^2 = \varphi(N^2) = O_n$, though $N^2 \neq O_n$.
   If n is odd, $(\varphi(A))^2 = \varphi(A^2) = O_n$, though $A^2 = A \neq O$.
   So we reach a contradiction in this case as well.
3. $a = 0,\ b = 1$. Since $N^n = O_n$, it must follow that $(\varphi(N))^n = O_n$ by Lemma 2.

Let $\varphi(N) = C$ and $v = (0,0,\dots,0,1,1)^T \in \mathbb{R}^n$. We obtain

$$Cv = (0,0,\dots,1,1,0)^T, \dots,\ C^{n-2}v = (1,1,\dots,0,0,0)^T,\ C^{n-1}v = (0,1,1,\dots,1,0)^T.$$

Since $C^n v = C\,(0,1,1,\dots,1,0)$ is equal to the sum of columns from the second to $(n-1)$-th, it cannot be the zero vector, because its first coordinate is $(a+(n-2)b)$ mod2, while the $(n-1)$-th coordinate is $(b+(n-2)b)$ mod2; obviously these two numbers cannot be equal to zero at the same time. Since $C^n v \neq 0$, we deduce that $C^n = (\varphi(N))^n \neq 0$, which is again a contradiction.

Hence, the only possible case is

4. $a = 1,\ b = 0$.

Now assume $p \neq 2$.

Let D denote the diagonal matrix $\begin{pmatrix} p-1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$.

Since $(p-1)^2 = 1$ in $\mathbb{Z}_p$, it is easy to see that $DAD = A$. Therefore, $\varphi(A) = D\varphi(A)D$, because $D \in GL_n(\mathbb{Z}_p)$. We have

$$D\varphi(A)D = \begin{pmatrix} a & (p-1)b & \cdots & (p-1)b & 0 \\ (p-1)b & a & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (p-1)b & b & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

so $(p-1)b = b$. We deduce that $b = 0$, as $p \neq 2$.

Thus, $\varphi(A) = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$.

Applying Lemma 2 one more time, we have $a \neq 0$. Once again applying Fermat's little theorem, we get:

$$\varphi(A) = \varphi(A^{p-1}) = \varphi(A)^{p-1} = A.$$

This means that $\varphi(N) = N$, which completes our proof. □

Proof of Theorem 2. The automorphisms of $GL_n(\mathbb{Z}_p)$ can be represented as a composition of automorphisms of three types: inner automorphisms, radial automorphisms and transpose-inverse automorphism [8,9].

Obviously the inner automorphisms of $GL_n(\mathbb{Z}_p)$ can be extended to those of $M_n(\mathbb{Z}_p)$. We show that the other types of isomorphisms do not possess that property.

We first consider the simplest case $n = 2,\ p = 2$. Since $GL_2(\mathbb{Z}_2)$ is not cyclic and contains 6 elements, it is isomorphic to S_3, which is a complete group, i.e. its every automorphism is inner. Hence τ is also surjective in this case, and $\mathrm{Aut}(M_2(\mathbb{Z}_2)) \cong \mathrm{Aut}(GL_2(\mathbb{Z}_2))$.

Let ψ denote the following automorphism of $GL_n(\mathbb{Z}_p)$:

$$\psi(A) = (A^T)^{-1}, \quad A \in GL_n(\mathbb{Z}_p).$$

Proposition. *If $n \geq 3$, there is no an automorphism of $M_n(\mathbb{Z}_p)$ such that its restriction to $GL_n(\mathbb{Z}_p)$ is equal to ψ. In other words, ψ has an empty preimage under τ. The same holds true is the case $n = 2$ if $p \neq 2$.*

Proof. Suppose to the contrary that the converse statement holds. Let us denote the extension of ψ to $M_n(\mathbb{Z}_p)$ by the same letter. Consider the following cases:

1. $n \geq 3$. Let $\psi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$.

Since $E_{11} = E_{11}P_{2k}$ and $(P_{2k}^T)^{-1} = P_{2k}$, we get

$$\psi(E_{11}) = \psi(E_{11}P_{2k}) = \psi(E_{11})\psi(P_{2k}) = \psi(E_{11})P_{2k}.$$

$$[\psi(E_{11})P_{2k}]_{i2} = [\psi(E_{11})]_{ik} = c_{ik}, \quad i = 1, 2, \ldots n, \quad \text{therefore,}$$

$$c_{i2} = c_{ik}, \quad i = 1,\dots,n, \quad k = 3,\dots,n.$$

On the other hand, $E_{11} = P_{2k}E_{11}$. Using the same argument, we get $c_{2j} = c_{kj}, \quad j = 1,\dots,n, \quad k = 3,\dots,n.$

Thus, $\psi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{12} \\ c_{21} & c_{22} & \cdots & c_{22} \\ \vdots & \vdots & \ddots & \vdots \\ c_{21} & c_{22} & \cdots & c_{22} \end{pmatrix}$.

Let L denote the lower triangular matrix with all ones below and on the main diagonal.

It is easy to show that $\psi(L) = (L^T)^{-1} = \begin{pmatrix} 1 & p-1 & 0 & \cdots & 0 \\ 0 & 1 & p-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & p-1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$.

Now, since $E_{11}L = E_{11}$, we have $\psi(E_{11})\psi(L) = \psi(E_{11})$.

$$\psi(E_{11})\psi(L) = \begin{pmatrix} c_{11} & (p-1)c_{11} + c_{12} & 0 & \cdots & 0 \\ c_{21} & (p-1)c_{21} + c_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{21} & (p-1)c_{21} + c_{22} & 0 & \cdots & 0 \end{pmatrix}.$$

It follows that $c_{12} - c_{11} = c_{12} = 0$ and $c_{22} - c_{21} = c_{22} = 0$, so we get $\psi(E_{11}) = O_n$, which contradicts the fact that ψ is an automorphism of $M_n(\mathbb{Z}_p)$. □

2. $n = 2$ and $p \neq 2$. Let $\psi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & p-1 \end{pmatrix}$.

Since $E_{11}D = E_{11}$ and $(D^T)^{-1} = D$, we obtain $\psi(E_{11}) = \psi(E_{11})D$. Consequently, $c_{12} = c_{22}$.

On the other hand, $E_{11}L = E_{11}$ and $(L^T)^{-1} = \begin{pmatrix} 1 & p-1 \\ 0 & 1 \end{pmatrix}$. Further calculations imply $c_{11} = c_{21} = 0$.

Thus $\psi(E_{11}) = O_n$, which leads to contradiction. □

Corollary. *The automorphism ψ defined above is non-inner automorphism of $GL_n(\mathbb{Z}_p)$ if $n \geq 3$ or $p \geq 3$.*

Proof. Indeed, if we had $\psi(A) = XAX^{-1}$ for some $X \in GL_n(\mathbb{Z}_p)$, its extension to $M_n(\mathbb{Z}_p)$ in the natural way would be an automorphism of $M_n(\mathbb{Z}_p)$, which is not possible due to Proposition 1. □

It can be easily shown that all the radial automorphisms are of the form

$$\chi(A) = (\det(A))^k A, \quad A \in GL_n(\mathbb{Z}_p).$$

Suppose χ extends to the automorphism of $M_n(\mathbb{Z}_p)$. We will denote it by the same letter.

If $p = 2$, this automorphism is simply the identity automorphism. So we can assume $p > 2$.

Let $\chi(E_{11}) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$.

Let I_n be the $n \times n$ identity matrix.

For $1 \le i < j \le n$ we have

$$\chi(E_{11}) = \chi((I_n + E_{ij})E_{11}) = \chi(I_n + E_{ij})\chi(E_{11}) = (I_n + E_{ij})\chi(E_{11}),$$

which implies $c_{ik} = 0, \;\; i = 2, \dots, n, \;\; k = 1, \dots, n$.

Analogously, for $1 \le j < i \le n$

$$\chi(E_{11}) = \chi(E_{11}(I_n + E_{ij})) = \chi(E_{11})\chi(I_n + E_{ij}) = \chi(E_{11})(I_n + E_{ij})$$

and hence $c_{kj} = 0, \;\; j = 2, \dots, n, \;\; k = 1, \dots, n$.

If χ is not identity, then there exists $A \in GL_n(\mathbb{Z}_p)$ such that $(\det(A))^k \ne 1$. Obviously, we can take A to be diagonal; moreover, let $[A]_{11} = 1$.

Then $\chi(E_{11}A) = \chi(E_{11})$. On the other hand, $\chi(E_{11}A) = \chi(E_{11})(\det(A))^k A$.

It follows that $c_{11}(\det(A))^k = c_{11}$, hence $c_{11} = 0$.

Thus, we find out that the radial automorphisms can not be extended to automorphisms of $M_n(\mathbb{Z}_p)$, which concludes our proof. □

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REFERENCES

1. Plotkin B. *Seven Lectures on the Universal Algebraic Geometry.* Preprint (2002). https://doi.org/10.48550/arXiv.math/0204245
2. Formanek E. A Question of B. Plotkin about the Semigroup of Endomorphisms of a Free Group. *Proc. Amer. Math. Soc.* **130** (2002), 935–937. https://doi.org/10.2307/2699537
3. Atabekyan V.S. The Automorphisms of Endomorphism Semigroups of Free Burnside Groups. *Int. J. Algebra Comput.* **25** (2015), 669–674. https://doi.org/10.1142/S0218196715500149
4. Atabekyan V.S., Aslanyan H.T. The Automorphisms of Endomorphism Semigroups of Relatively Free Groups. *Int. J. Algebra Comput.* **28** (2018), 207–215. https://doi.org/10.1142/S0218196718500108
5. Gluskin L.M. Automorphisms of Multiplicative Semigroups of Matrix Algebras. *Uspehi Mat. Nauk (N.S.)* **11** (1956), 199–206 (in Russian).
6. Halezov E.A. Automorphisms of Matrix Subgroups. *Dokl. Akad. Nauk SSSR* **96** (1954), 245–248 (in Russian).
7. Waterhouse W.C. Two Generators for the General Linear Groups over Finite Fields. *Linear Multilinear Algebra* **24** (1988), 227–230. https://doi.org/10.1080/03081088908817916

8. Dieudonne J. On the Automorphisms of the Classical Groups. *Mem. Amer. Math. Soc.* **2** (1951).
https://doi.org/10.1090/S0002-9939-1951-0040426-0
9. Waterhouse W.C. Automorphisms of $GL_n(R)$. *Proc. Am. Math. Soc.* **79** (1980), 347–351.
https://doi.org/10.2307/2043063

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ՎԵՐՋԱՎՈՐ ԷԼԵՄԵՆՏԱՐ ԱԲԵԼՅԱՆ ԽՄԲԵՐԻ ԷՆԴՈՄՈՐՖԻԶՄՆԵՐԻ ԿԻՍԱԽՄԲԵՐԻ ԱՎՏՈՄՈՐՖԻԶՄՆԵՐԻ ԽՄԲԵՐԻ ՄԱՍԻՆ

Հոդվածում ուսումնասիրվում են վերջավոր էլեմենտար աբելյան խմբերի էնդոմորֆիզմների կիսախմբերի և ավտոմորֆիզմների խմբերի ավտոմորֆիզմները: Մասնավորապես՝ մատրիցային գործողությունների վրա հիմնված պարզագույն մոտեցմամբ մենք ապացուցում ենք, որ $\mathrm{Aut}(\mathrm{End}(\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p))$ խումբը կարելի է կանոնական ներդնել $\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p))$ խմբի մեջ: Մենք նաև ցույց ենք տալիս, որ $\mathrm{End}(\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p)$-ի բոլոր ավտոմորֆիզմները ներքին են:

А. А. БАЙРАМЯН

О ГРУППАХ АВТОМОРФИЗМОВ ПОЛУГРУПП ЭНДОМОРФИЗМОВ КОНЕЧНЫХ ЭЛЕМЕНТАРНЫХ АБЕЛЕВЫХ ГРУПП

В статье изучаются автоморфизмы полугрупп эндоморфизмов и групп автоморфизмов конечных элементарных абелевых групп. В частности, используя элементарный подход, основанный на матричных операциях, мы доказываем, что $\mathrm{Aut}(\mathrm{End}(\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p))$ можно канонически вложить в $\mathrm{Aut}(\mathrm{Aut}(\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p))$. Мы также показываем, что все автоморфизмы $\mathrm{End}(\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\cdots\oplus\mathbb{Z}_p)$ являются внутренними.