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Mathematics

ON A GENERALIZED FORMULA OF TAYLOR–MACLAURIN TYPE IN A NEIGHBORHOOD OF +∞

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In the paper we consider systems of operators generated by Weil integral and derivative, and functions generated by exponential type functions. For a certain class of functions a generalization of Taylor–Maclaurin type formula is obtained in a neighborhood of $+\infty$.

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Preliminaries. Let $\alpha \in (0,+\infty)$, $f(x) \in L(0,l)$ $(0 < l < +\infty)$. The function $D_\infty^{-\alpha} \equiv \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_x^{+\infty} (t-x)^{\alpha-1} f(t)dt$ is called the Weil integral of order α of function $f(x)$.

Let $\alpha \in [0,1)$, $1-\alpha = 1/\rho$ $(\rho \geq 1)$, $f(x) \in L(0,l)$. The function $D_\infty^{1/\rho} f(x) \equiv \equiv \dfrac{d}{dx} D_\infty^{-\alpha} f(x)$ is called the $1/\rho$ order Weil derivative of $f(x)$.

The operators $D_\infty^0 f(x) \equiv f(x)$, $D_\infty^{1/\rho} f(x)$, $D_\infty^{n/\rho} f(x) \equiv D_\infty^{1/\rho}\left(D_\infty^{n-1/\rho} f(x)\right)$, $n = 0,1,\ldots$ are called Weil operators of successive differentiation of $f(x)$ of order n/ρ.

In [2] the following classes of function are introduced:

- $C_\alpha^{(\infty)}$ $(0 \leq \alpha < 1)$ is the class of function $f(x)$ possessing all successive derivatives $D_{+\infty}^n f(x) \equiv f^{(n)}(x)$, $n = 0,1,\ldots,$ satisfying

$$\sup_{0 \leq x < \infty} \left|(1+x^{\alpha m}) f^{(n)}(x)\right| < +\infty, \qquad n,m = 0,1,2,\ldots;$$

- $C_\alpha^{+(\infty)}$ $(0 \leq \alpha < 1)$ is the class of functions possessing all successive Weil derivatives $D_\infty^{n/\rho} f(x)$, $n = 0,1,\ldots,$ that are continuous on $[0,+\infty)$ and satisfy

$$\sup_{0 \leq x < \infty} \left|(1+x^{\alpha m}) D_\infty^{n/\rho} f(x)\right| < +\infty, \qquad n,m = 0,1,2,\ldots$$

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In [2] it is proved that these classes coincide.

1.1. The Mittag–Leffler type function $E_\rho = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n\rho^{-1})}$, $\rho > 0$ is an entire function of order ρ and type 1 for any value of parameter μ ([1], Ch.VI, §1).

1.2. For any $\mu > 0, \alpha > 0$ the following formula holds:

$$\frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} E_\rho(\lambda \xi^{1/\rho};\mu)\xi^{\mu-1} d\xi = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho};\mu+\alpha), \quad (1.1)$$

where λ is a complex parameter, and the integration is taken along the line segment connecting points 0 and z ([1], Ch. III, (1.16)).

1.3. For any $\alpha > 0, \beta > 0$ the following formula holds:

$$\int_0^l x^{\alpha-1} E_\rho(\lambda x^{1/\rho};\alpha)(1-x)^{\beta-1} E_\rho(\lambda^*(l-x)^{1/\rho};\beta)dx =$$

$$= \frac{\lambda E_\rho(l^{1/\rho}\lambda;\alpha+\beta) - \lambda^* E_\rho(l^{1/\rho}\lambda^*;\alpha+\beta)}{\lambda - \lambda^*} l^{\alpha+\beta-1}, \quad (1.2)$$

where λ and λ^* are any complex parameters ([1], Ch.III (1.21), [4]).

From (1.2), using the famous $E_\rho(z;\ 2/\rho) = \frac{1}{z}\left\{E_\rho(z;1/\rho) - 1/\Gamma(1/\rho)\right\}$ formula, for the particular case $\alpha = \beta = 1/\rho$, we obtain

$$\int_0^x E_\rho(\lambda t^{1/\rho};1/\rho)t^{1/\rho-1} E_\rho(\lambda^*(x-t)^{1/\rho};1/\rho)(x-t)^{1/\rho-1}dt =$$

$$= (E_\rho(\lambda x^{1/\rho};1/\rho) - E_\rho(\lambda^* x^{1/\rho};1/\rho))x^{1/\rho-1}/(\lambda - \lambda^*). \quad (1.3)$$

1.4. It is known (see [2], (2.22)) that the function $\tilde{e}_\rho(x;\lambda) = e^{-\lambda^\rho x}$, is the solution of the following Cauchy type problem: $D_\infty^{1/\rho} y(x) + \lambda y(x) = 0,\ y(0) = 1$. Hence,

$$(D_\infty^{1/\rho} + \lambda^*)e^{-\lambda^\rho x} = (\lambda^* - \lambda)e^{-\lambda^\rho x}. \quad (1.4)$$

2. The Main Result. Let $\rho \geq 1$ $(1-\alpha = 1/\rho)$, and $\left\{\lambda_j\right\}_0^\infty$ be an arbitrary increasing sequence of positive numbers. We consider the following systems of operators $\left\{L_\infty^{n/\rho}\right\}_0^\infty, \left\{\tilde{L}_\infty^{n/\rho}\right\}_0^\infty$, and functions $\{\varphi_n(x)\}_0^\infty$, $x \in [0,+\infty)$:

$$L_\infty^{0/\rho} f \equiv f,\ L_\infty^{n/\rho} f = \prod_{j=0}^{n-1}\left(D_\infty^{1/\rho} + \lambda_j\right) f,\ n \geq 1,\ \tilde{L}_\infty^{n/\rho} f = e^{-\lambda^\rho x} L_\infty^{n/\rho} f,\ n \geq 0, \quad (2.1)$$

$$\varphi_0(x) = e^{-\lambda_0^\rho x},\ \varphi_n(x) = \sum_{k=0}^{n} C_k^{(n)} e^{-\lambda_k^\rho x}, n \geq 1,\ C_k^{(n)} = \left\{\prod_{j=0; j\neq k}^{n} (\lambda_j - \lambda_k)\right\}^{-1}. \quad (2.2)$$

We note that the systems of operators $\left\{L_\infty^{n/\rho}\right\}_0^\infty$ and functions $\{\varphi_n(x)\}_0^\infty$ were first introduced in [3].

L e m m a 2 . 1 .

1°. $L_\infty^{k/\rho}\{\varphi_n(x)\} \equiv \tilde{L}_\infty^{k/\rho}\{\varphi_n(x)\} \equiv 0,\ \forall n \geq 0,\ k \geq n+1,\ x \in [0,+\infty). \quad (2.3)$

2°. $\tilde{L}_\infty^{n/\rho}\{\varphi_n(x)\} \equiv 1,\ \forall n \geq 0,\ x \in [0,+\infty). \quad (2.4)$

$$3^\circ.\ \lim_{x\to+\infty}\tilde{L}_\infty^{k/\rho}\{\varphi_n(x)\}=\left\{\prod_{j=k+1}^{n}(\lambda_j-\lambda_k)\right\}^{-1},\forall n\geq 1,\ 0\leq k\leq n-1. \quad (2.5)$$

Proof. We note that according to (2.1), (2.2), by using (1.4), we will have

$$1^\circ.\ \tilde{L}_\infty^{(n+1)/\rho}\{\varphi_n(x)\}=e^{\lambda_{n+1}^\rho x}L_\infty^{(n+1)/\rho}=e^{\lambda_{n+1}^\rho x}\sum_{i=0}^{n}C_i^{(n)}L_\infty^{n/\rho}\left\{e^{-\lambda_i^\rho x}\right\}=$$

$$=e^{\lambda_{n+1}^\rho x}\sum_{i=0}^{n}C_i^{(n)}\prod_{j=0}^{n}(D_\infty^{1/\rho}+\lambda_j)e^{-\lambda_i^\rho x}=e^{\lambda_{n+1}^\rho x}\sum_{i=0}^{n}C_i^{(n)}\prod_{j=0}^{n}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}. \quad (2.6)$$

But since $C_i^{(n)}\prod_{j=0}^{n}(\lambda_j-\lambda_i)=0,\ i=0,1,\ldots,n,$ then from (2.6) we will obtain $\tilde{L}_\infty^{(n+1)/\rho}\{\varphi_n(x)\}\equiv 0,\ n\geq 0.$

It is obvious that $\tilde{L}_\infty^{k/\rho}\{\varphi_n(x)\}\equiv 0,$ when $k>n+1.$

$$2^\circ.\ \tilde{L}_\infty^{n/\rho}\{\varphi_n(x)\}=e^{\lambda_n^\rho x}\sum_{i=0}^{n}C_i^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}=$$

$$=e^{\lambda_n^\rho x}C_n^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_n)e^{-\lambda_n^\rho x}=C_n^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_n)=1,$$

since $C_n^{(n)}=\left\{\prod_{j=0,j\neq n}^{n-1}(\lambda_j-\lambda_n)\right\}^{-1}.$

3°. Let $0\leq k\leq n-1.$

$$\tilde{L}_\infty^{k/\rho}\{\varphi_n(x)\}=e^{\lambda_k^\rho x}\sum_{i=0}^{n}C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}=e^{\lambda_k^\rho x}\sum_{i=k}^{n}C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)e^{-\lambda_i^\rho x}=$$

$$=C_k^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_k)+\sum_{i=k+1}^{n}C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)e^{-(\lambda_i^\rho-\lambda_k^\rho)x}. \quad (2.7)$$

So, we get $\lim_{x\to+\infty}\tilde{L}_\infty^{k/\rho}\{\varphi_n(x)\}=C_k^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_k)=\left\{\prod_{j=k+1}^{n}(\lambda_j-\lambda_k)\right\}^{-1}.$ □

Lemma 2.2. Let

$\rho\geq 1\ \ (1-\alpha)=1/\rho,\quad f(x)\in C_\alpha^{+(\infty)},\quad e^{\lambda_0^\rho x}|f(x)|\in L(0,+\infty).$

Then the function

$$y(x)=-\int_x^{+\infty}e_\rho(t-x;\lambda_0)e^{-\lambda_1^\rho t}f(t)dt \quad (2.8)$$

is the solution of the following Cauchy type problem:

$$\tilde{L}_\infty^{1/\rho}y=f(x),\qquad \tilde{L}_\infty^{0/\rho}y\big|_{x=+\infty}=0,\qquad \text{where } 0<\lambda_0<\lambda_1. \quad (2.9)$$

Proof. First note that the integral on the right-hand side of (2.8) converges absolutely and uniformly on $x\in(0,+\infty)$ according to Lemma 2.1 (see [3]). Using the definition (2.1) of $\tilde{L}_\infty^{k/\rho}(k=1),$ we have $\tilde{L}_\infty^{1/\rho}y=e^{\lambda_1^\rho x}L_\infty^{1/\rho}y=f(x),$ implying $L_\infty^{1/\rho}y=e^{-\lambda_1^\rho x}f(x)$, i.e.

$$\left(D_{\infty}^{1/\rho}+\lambda_0\right)y=e^{-\lambda_1^{\rho}x}f(x). \tag{2.10}$$

Due to Lemma 2.2 (see [3]), we have $y=-\int_x^{+\infty}e_\rho(t-x;\lambda_0)e^{-\lambda_1^{\rho}t}f(t)dt$, where $e_\rho(x;\lambda_0)\equiv E_\rho\left(\lambda_0 x^{\frac{1}{\rho}};\frac{1}{\rho}\right)x^{\frac{1}{\rho}-1}$.

It is easy to see that the function

$$\tilde{y}(x)=c_0e^{-\lambda_0^{\rho}x}-\int_x^{+\infty}e_\rho(t-x;\lambda_0)e^{-\lambda_1^{\rho}t}f(t)dt, \tag{2.11}$$

also solves (2.9).

Now, using $\tilde{L}_{\infty}^{0/\rho}\tilde{y}(x)\big|_{x=+\infty}=0$, we show that $c_0=0$.

Indeed, by the definition of $\tilde{L}_{\infty}^{n/\rho}$ $(n=0)$, from (2.11) we get

$$\tilde{L}_{\infty}^{0/\rho}\tilde{y}(x)=e^{\lambda_0^{\rho}x}\tilde{y}(x)=c_0-e^{\lambda_0^{\rho}x}\int_x^{+\infty}e_\rho(t-x;\lambda_0)e^{-\lambda_1^{\rho}t}f(t)dt. \tag{2.12}$$

Next we show that $e^{\lambda_0^{\rho}x}\int_x^{+\infty}e_\rho(t-x;\lambda_0)e^{-\lambda_1^{\rho}t}f(t)dt\to 0$, as $x\to+\infty$. Namely, $\left|e^{\lambda_0^{\rho}x}\int_x^{+\infty}e_\rho(t-x;\lambda_0)e^{-\lambda_1^{\rho}t}f(t)dt\right|\le$

$$\le e^{-(\lambda_1^{\rho}-\lambda_0^{\rho})x}\int_x^{+\infty}\left|e_\rho(t-x;\lambda_0)e^{-\lambda_1^{\rho}t}f(t)\right|dt\to 0, \text{ as } x\to+\infty.$$

So, obviously, $\lim\limits_{x\to+\infty}\tilde{L}_{\infty}^{0/\rho}\tilde{y}(x)=c_0=0$. □

L e m m a 2 . 3 . Let
$\rho\ge 1$ $(1-\alpha)=1/\rho$, $f(x)\in C_\alpha^{+(\infty)}$, $e^{\lambda_1^{\rho}x}|f(x)|\in L(0,+\infty)$.
Then the function

$$y(x)=\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^{\rho}t_1}f(t_1)dt_1 \tag{2.13}$$

is the solution of the following Cauchy type problem:

$$\tilde{L}_{\infty}^{2/\rho}y=f(x),\ \ \tilde{L}_{\infty}^{k/\rho}y|_{x=+\infty}=0,(k=0,1), \text{ where } 0<\lambda_0<\lambda_1<\lambda_2. \tag{2.14}$$

Proof. Due to the definition of operator $\tilde{L}_{\infty}^{2/\rho}$, we have

$$\tilde{L}_{\infty}^{2/\rho}y=e^{\lambda_2^{\rho}x}L_{\infty}^{2/\rho}y=e^{\lambda_2^{\rho}x}\left(D_{\infty}^{1/\rho}+\lambda_1\right)\left(D_{\infty}^{1/\rho}+\lambda_0\right)y=f(x). \tag{2.15}$$

That is

$$\left(D_{\infty}^{1/\rho}+\lambda_1\right)\left(D_{\infty}^{1/\rho}+\lambda_0\right)y=e^{-\lambda_2^{\rho}x}f(x).$$

Now, using Lemma 3.2 (see [3]), we have

$$y(x)=\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^{\rho}t_1}f(t_1)dt_1. \tag{2.16}$$

We note that the function

$$\tilde{y}(x)=c_0e^{-\lambda_0^\rho x}+c_1e^{-\lambda_1^\rho x}+$$
$$+\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1 \tag{2.17}$$

also solves the equation (2.14), since

$$\left(D_\infty^{1/\rho}+\lambda_0\right)\left(D_\infty^{1/\rho}+\lambda_1\right)\left\{c_0e^{-\lambda_0^\rho x}+c_1e^{-\lambda_1^\rho x}\right\}\equiv 0.$$

We show that $c_0=c_1=0$. Note that

$$L_\infty^{0/\rho}\tilde{y}(x)=c_0+c_1e^{-(\lambda_1^\rho-\lambda_0^\rho)x}+$$
$$+e^{\lambda_0^\rho x}\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1. \tag{2.18}$$

We will prove that the integral on the right-hand side of (2.18) tends to zero, as $x\to+\infty$. It is known (see [3], proof of the Lemma 3.2) that

$$e^{\lambda_0^\rho x}\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1=$$
$$=\frac{e^{\lambda_0^\rho x}}{\lambda_1-\lambda_0}\left\{\int_x^{+\infty}e_\rho(\tau-x;\lambda_1)e^{-\lambda_2^\rho\tau}f(\tau)d\tau-\int_x^{+\infty}e_\rho(\tau-x;\lambda_0)e^{-\lambda_2^\rho\tau}f(\tau)d\tau\right\}.$$

So we obtain

$$\left|e^{\lambda_0^\rho x}\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1\right|\le$$
$$\le\frac{e^{-(\lambda_2^\rho-\lambda_0^\rho)x}}{\lambda_1-\lambda_0}\left\{\int_x^{+\infty}\left|e_\rho(\tau-x;\lambda_1)f(\tau)\right|d\tau+\int_x^{+\infty}\left|e_\rho(\tau-x;\lambda_0)f(\tau)\right|d\tau\right\}\to 0,$$

as $x\to+\infty$.

Now, using (2.18) we deduce that $\lim\limits_{x\to+\infty}\tilde{L}_\infty^{0/\rho}\tilde{y}(x)=c_0=0$.

Next, we apply the operator $\tilde{L}_\infty^{1/\rho}$ to (2.18):

$$\tilde{L}_\infty^{1/\rho}\tilde{y}(x)=e^{\lambda_1^\rho x}\Big\{c_1\left(D_\infty^{1/\rho}+\lambda_0\right)e^{-\lambda_1^\rho x}+\left(D_\infty^{1/\rho}+\lambda_0\right)\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\times$$
$$\times\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1\Big\}=c_1(\lambda_0-\lambda_1)+ \tag{2.19}$$
$$+e^{\lambda_1^\rho t_1}\left(D_\infty^{1/\rho}+\lambda_0\right)\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1.$$

Note that, according to Lemma 2.2 (see [3]), we have

$$\left(D_\infty^{1/\rho}+\lambda_0\right)\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1=$$
$$=-\int_x^{+\infty}e_\rho(t_1-x;\lambda_1)e^{-\lambda_2^\rho t_1}f(t_1)dt_1 \tag{2.20}$$

Then, from (2.19) by using (2.20), we obtain

$$\tilde{L}_\infty^{1/\rho}\tilde{y}(x) = c_1(\lambda_0-\lambda_1) - e^{\lambda_1^\rho x}\int_x^{+\infty} e_\rho(t_1-x;\lambda_1)e^{-\lambda_2^\rho t_1} f(t_1)dt_1. \qquad (2.21)$$

It is easy to see that $\left| e^{\lambda_1^\rho x}\int_x^{+\infty} e_\rho(t_1-x;\lambda_1)e^{-\lambda_2^\rho t_1} f(t_1)dt_1\right| \leq$

$$\leq e^{-(\lambda_2^\rho-\lambda_1^\rho)x}\int_x^{+\infty}\left| e_\rho(t_1-x;\lambda_1)e^{-\lambda_2^\rho t_1} f(t_1)\right| dt_1 \to 0 \quad \text{as } x\to+\infty.$$

So, from (2.21) we get $\lim\limits_{x\to+\infty}\tilde{L}_\infty^{1/\rho}\tilde{y}(x) = c_1(\lambda_0-\lambda_1)=0,\ c_1=0.$ □

L e m m a 2 . 4 . Let

$\rho\geq 1\ \ (1-\alpha)=1/\rho,\ f(x)\in C_\alpha^{+(\infty)},\ e^{\lambda_{n-1}^\rho x}|f(x)|\in L(0,+\infty),\ n\geq 1.$

Then the function

$$y(x) = (-1)^n\int_x^{+\infty} e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty} e_\rho(t_1-t_0;\lambda_1)dt_1\times\cdots\times$$

$$\times\int_{t_{n-2}}^{+\infty} e_\rho(t_{n-1}-t_{n-2};\lambda_{n-1})e^{-\lambda_n^\rho t_{n-1}}dt_{n-1} \qquad (2.22)$$

is the solution of the following Cauchy type problem:

$$\tilde{L}_\infty^{n/\rho}\tilde{y}(x) = f(x), \qquad \tilde{L}_\infty^{k/\rho}y|_{x=+\infty}=0, \qquad k=0,1,\ldots,n-1. \qquad (2.23)$$

We omit the proof of Lemma 2.4 can be done in the same as for Lemmas 2.2, 2.3.

L e m m a 2 . 5 . Let $P_n(x)=\sum\limits_{k=0}^{n} a_k\varphi_k(x),\ x\in[0,+\infty)$. Then the coefficients $\{a_k\}_0^n$ can be determined from formulae

$$\left[\begin{array}{l}\lim\limits_{x\to+\infty}\tilde{L}_\infty^{i/\rho}P_n(x)\equiv\tilde{L}_\infty^{i/\rho}P_n(+\infty) = a_i+\sum\limits_{k=i+1}^{n} a_k\left\{\prod\limits_{j=i+1}^{k}(\lambda_j-\lambda_i)\right\}^{-1}, \\ a_n=\tilde{L}_\infty^{n/\rho}P_n(+\infty).\end{array}\right. \qquad (2.24)$$

Proof. Let $i=n$. We apply the operator $\tilde{L}_\infty^{n/\rho}$ to the function $P_n(x)$. Then, using the formulas (2.3), (2.4), we get

$$\lim_{x\to+\infty}\tilde{L}_\infty^{n/\rho}P_n(x)\equiv\tilde{L}_\infty^{n/\rho}P_n(+\infty) = \lim_{x\to+\infty}\sum_{k=0}^{n} a_k\tilde{L}_\infty^{n/\rho}\{\varphi_k(x)\} = a_n.$$

Now assume $0\leq i\leq n-1$. We apply the operator $\tilde{L}_\infty^{i/\rho}$ to the function $P_n(x)$:

$$\tilde{L}_\infty^{i/\rho}P_n(x)\equiv\sum_{k=0}^{n} a_k\tilde{L}_\infty^{i/\rho}\{\varphi_k(x)\} = a_i+\sum_{k=i+1}^{n} a_k\tilde{L}_\infty^{i/\rho}\{\varphi_k(x)\}. \qquad (2.25)$$

But according to (2.5),

$$\lim_{x\to+\infty}\tilde{L}_\infty^{i/\rho}\{\varphi_k(x)\} = \left\{\prod_{j=i+1}^{k}(\lambda_j-\lambda_i)\right\}^{-1}. \qquad (2.26)$$

and from (2.25) we can easily obtain that

$$\lim_{x\to+\infty}\tilde{L}_{\infty}^{i/\rho}P_n(x)=a_i+\sum_{k=i+1}^{n}a_k\left\{\prod_{j=i+1}^{k}(\lambda_j-\lambda_i)\right\}^{-1}. \qquad \square$$

Let $\rho\geq 1$ $(1-\alpha=1/\rho)$. Denote by $\tilde{C}_{\alpha}^{+(\infty)}$ the set functions $f(x)$ satisfying the following conditions:

1. $f(x)\in\tilde{C}_{\alpha}^{+(\infty)}$;
2. $e^{\lambda_n^{\rho}x}\left|L_{\infty}^{(n+1)/\rho}f(x)\right|\in L(0,+\infty),\ n=0,1,\dots$;
3. $\lim\limits_{x\to+\infty}\tilde{L}_{\infty}^{n/\rho}f(x)=\tilde{L}_{\infty}^{n/\rho}f(+\infty)<+\infty,\ n=0,1,\dots$

It can be easily seen that the functions $f_n(x)=e^{-\lambda_n^{\rho}x},\ \varphi_n(x)=\sum\limits_{k=0}^{n}C_k^{(n)}e^{-\lambda_k^{\rho}x}$, $P_n(x)=\sum\limits_{k=0}^{n}a_k\varphi_k(x),\ n=0,1,\dots,$ belong to the class $\tilde{C}_{\alpha}^{+(\infty)}$.

Theorem 2.1. If $f(x)\in\tilde{C}_{\alpha}^{+(\infty)}$ then the following formula is true:

$$f(x)=\sum_{k=0}^{n}a_k\varphi_k(x)+R_n(x),x\in[0,+\infty)\quad\forall n\geq 0 \tag{2.27}$$

where the coefficients $\{a_k\}_0^n$ are determined from formulae

$$\left[\begin{array}{l} a_n=\tilde{L}_{\infty}^{n/\rho}f(+\infty),\\ \lim\limits_{x\to+\infty}\tilde{L}_{\infty}^{i/\rho}f(x)=\tilde{L}_{\infty}^{i/\rho}f(+\infty)=a_i+\sum\limits_{k=i+1}^{n}a_k\left\{\prod\limits_{j=i+1}^{k}(\lambda_j-\lambda_i)\right\}^{-1}. \end{array}\right. \tag{2.28}$$

$$R_n(x)=(-1)^{n+1}\int_x^{+\infty}K_{\rho}(\tau-x;\{\lambda_j\}_0^n)L_{\infty}^{(n+1)/\rho}f(\tau)d\tau, \tag{2.29}$$

where $K_{\rho}(x;\{\lambda_j\}_0^n)=\sum\limits_{i=0}^{n}\left\{\prod\limits_{j=0,j\neq 1}^{n}(\lambda_j-\lambda_i)\right\}^{-1}e_{\rho}(x;\lambda_i)$.

Proof. Denote $P_n(x)=\sum\limits_{k=0}^{n}a_k\varphi_k(x),\ R_n(x)=f(x)-P_n(x)$, where $\{a_k\}_0^n$ are determined from (2.28). We note that $\tilde{L}_{\infty}^{k/\rho}P_n(+\infty)=\tilde{L}_{\infty}^{k/\rho}f(+\infty),\ k=0,1,\dots,n$. So, $\tilde{L}_{\infty}^{k/\rho}R_n(x)|_{x=+\infty}=0,\ k=0,1,\dots,n$ and $\tilde{L}_{\infty}^{(n+1)/\rho}R_n(x)=\tilde{L}_{\infty}^{(n+1)/\rho}f(x)$.

We note that the function $R_n(x)$ is the solution of the following Cauchy type problem: $\tilde{L}_{\infty}^{(n+1)/\rho}R_n(x)=\tilde{L}_{\infty}^{(n+1)/\rho}f(x),\ \tilde{L}_{\infty}^{(k)/\rho}R_n(x)|_{x\to+\infty}=0,\quad k=0,1,\dots,n$. According to Lemma 2.4, we get

$$\begin{aligned} R_n(x)&=(-1)^{n+1}\int_x^{+\infty}e_{\rho}(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_{\rho}(t_1-t_0;\lambda_1)dt_1\times\cdots\times\\ &\times\int_{t_{n-1}}^{+\infty}e_{\rho}(t_n-t_{n-1};\lambda_n)e^{-\lambda_{n+1}^{\rho}t_n}\tilde{L}_{\infty}^{(n+1)/\rho}f(t_n)dt_n\\ &=(-1)^{n+1}\int_x^{+\infty}e_{\rho}(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_{\rho}(t_1-t_0;\lambda_1)dt_1\times\cdots\times\\ &\times\int_{t_{n-1}}^{+\infty}e_{\rho}(t_n-t_{n-1};\lambda_n)L_{\infty}^{(n+1)/\rho}f(t_n)dt_n. \end{aligned} \tag{2.30}$$

Further, in the light of Lemma 2.3 (see [3]), we obtain (2.29), where

$$K_n(x;\{\lambda_j\}_0^n)=\sum_{i=0}^{n}\left\{\prod_{j=0,j\neq i}^{n}(\lambda_j-\lambda_i)\right\}^{-1}e_\rho(x;\lambda_i), e_\rho(x;\lambda_i)=E_\rho\left(\lambda_i x^{\frac{1}{\rho}};\frac{1}{\rho}\right)x^{\frac{1}{\rho}-1}. \square$$

Theorem 2.1′. Let $\rho\geq 1$ $(1-\alpha=1/\rho)$, $\lim\limits_{n\to\infty}\lambda_n=\tilde{\lambda}<+\infty$. Then for any $n\geq 0$ the following formula is true:

$$e^{-\tilde{\lambda}^\rho x}=(-1)^{n+1}\int_x^{+\infty}e_\rho(t_0-x;\lambda_0)dt_0\int_{t_0}^{+\infty}e_\rho(t_1-t_0;\lambda_1)dt_1\times\cdots\times$$

$$\times\int_{t_{n-1}}^{+\infty}e_\rho(t_n-t_{n-1};\lambda_n)L_\infty^{(n+1)/\rho}\{e^{-\tilde{\lambda}^\rho t_n}\}dt_n. \qquad (2.31)$$

Proof. We show that $e^{-\tilde{\lambda}^\rho x}\in\tilde{C}_\alpha^{+(\infty)}$. It is obvious that $D_\infty^{n/\rho}\{e^{-\tilde{\lambda}^\rho x}\}=$ $=(-\tilde{\lambda})^n e^{-\tilde{\lambda}^\rho x}, n=0,1,\ldots$ Consequently,

$$\sup_{0\leq x<\infty}\left|(1+x^{ma}(\tilde{\lambda})^n)e^{-\tilde{\lambda}^\rho x}\right|<+\infty,\ n,m=0,1\ldots,\text{ i.e. } e^{-\tilde{\lambda}^\rho x}\in\tilde{C}_\alpha^{+(\infty)}.$$

Further,

$$e^{\lambda^\rho x}L_\infty^{(n+1)/\rho}\{e^{-\tilde{\lambda}^\rho x}\}=e^{\lambda_n^\rho x}\prod_{j=0}^{n}(D_\infty^{1/\rho}+\lambda_j)e^{-\tilde{\lambda}^\rho x}=$$

$$=\prod_{j=0}^{n}(\lambda_j-\tilde{\lambda})e^{-(\tilde{\lambda}^\rho-\tilde{\lambda}_n^\rho)x}\in L(0;+\infty),\quad n=0,1,\ldots$$

We note that

$$\lim_{x\to+\infty}L_\infty^{0/\rho}\{e^{-\tilde{\lambda}^\rho x}\}=\lim_{x\to+\infty}e^{-(\tilde{\lambda}^\rho-\tilde{\lambda}_n^\rho)x}=0,\qquad\text{and for}\qquad n\geq 1$$

$$\lim_{x\to+\infty}\tilde{L}_\infty^{n/\rho}\{e^{-\tilde{\lambda}^\rho x}\}=\lim_{x\to+\infty}\prod_{j=0}^{n}(\lambda_j-\tilde{\lambda})e^{-(\tilde{\lambda}^\rho-\tilde{\lambda}_n^\rho)x}=0,$$

Since $f(x)=e^{-\tilde{\lambda}^\rho x}\in\tilde{C}_\alpha^{+(\infty)}$, then using Theorem 2.1, we obtain (2.31) □

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