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Mathematics

DEFORMATION OF THE REAL PART OF β-UNIFORM ALGEBRA

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In this paper we investigate the deformation of the real part of β-uniform algebra on a locally compact Hausdorff space. We prove that if the deformation semigroup contains at least one deformation other than the affinity, then β-uniform algebra coincides with the algebra of all complex-valued bounded continuous functions.

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1. Let Ω be locally compact Hausdorff space, which admits a compact exhaustion (i.e. $\Omega = \bigcup\limits_{n=1}^{\infty} K_n$, where $K_n \subset K_{n+1}$, and every $K_n \subset \Omega$ is a compact set) and $C_0(\Omega)$ is the algebra of all continuous complex-valued functions on Ω vanishing at "infinity". On the algebra $C_b(\Omega)$ of all bounded continuous complex-valued functions on Ω we define a topology by the family of semi-norms $\{P_g\}_{g\in C_0(\Omega)}$, where $P_g(f) = \sup\limits_{\Omega}|fg|$, for each $f \in C_b(\Omega)$. The topology on $C_b(\Omega)$, given by the family of semi-norms $\{P_g\}_{g\in C_0(\Omega)}$, is called β-uniform topology and the algebra $C_b(\Omega)$ equipped with this topology will be denoted by $C_\beta(\Omega)$. Recall that the closure of a subalgebra $\mathcal{A} \subset C_\beta(\Omega)$ in β-uniform topology is called *β-uniform,* if it contains the constants and separates the points of Ω, i.e. for any points $x_1, x_2 \in \Omega$, where $x_1 \neq x_2$, there exists a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$ [1–3].

Since the uniform topology is stronger then the β-uniform topology, β-uniform algebra is closed in the *sup*-norm of a subalgebra of the uniform algebra $C_b(\Omega)$. Algebra $\mathcal{A}$, endowed with the uniform topology, is denoted, as in [3], by $\mathcal{A}_\infty$, and its maximal ideal space is denoted by $M_{\mathcal{A}_\infty}$. By virtue of I.M. Gelfand theory [4–6], each multiplicative functional on the algebra $\mathcal{A}_\infty$ is continuous in the uniform norm, which is a cult in this theory, and $M_{\mathcal{A}_\infty}$ is a compact $*$-weak topology of the subset of the unit ball in $\mathcal{A}^*_\infty$-space conjugate to $\mathcal{A}_\infty$.

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Recall (see [3]) that the Stone–Cech $\mathcal{A}$-*compactification* for Ω is the closure of $*$-weak topology of Ω in the maximal ideal space $M_{\mathcal{A}_\infty}$ for uniform algebra $\mathcal{A}_\infty$.

We denote by $M_{\mathcal{A}}$ the set of all β-continuous linear multiplicative functionals of β-uniform algebra $\mathcal{A}$. It is clear that $M_{\mathcal{A}} \subset M_{\mathcal{A}_\infty}$. The Shilov boundary of the algebra $\mathcal{A}_\infty$ is denoted, as usual, via $\partial\mathcal{A}_\infty$, and the set $\partial\mathcal{A} = \partial\mathcal{A}_\infty \cap \Omega$ is called the β-*Shilov boundary* of $\mathcal{A}$ (this boundary can be empty).

Let Ω_0 be the Stone–Cech $\mathcal{A}$-compactification of Ω. Then the set Ω is dense in the compact set Ω_0, and each function $\mathcal{A}$ uniquely extends to a function on Ω_0. This extension gives rise to a uniform algebra $\mathcal{A}_{(0)}$ on Ω_0, so, $\mathcal{A}_{(0)}\big|_{\Omega} = \mathcal{A}$ [3, 7, 8].

2. Let $C_{\mathbb{R}}(\Omega)$ be the space of all real-valued continuous functions on Ω. Each real-valued function $\mathbf{h} \in C_{\mathbb{R}}(\Omega)$ induces a map $\Phi_{\mathbf{h}} : C_{\mathbb{R}}(\Omega) \to C_{\mathbb{R}}(\Omega)$, where $\Phi_h(f) = h(f)$. If $h(t) \in C_{\mathbb{R}}(\Omega)$ is an affine function, i.e. $h(t) = \alpha t + \beta$, where $\alpha, \beta \in \mathbb{R}$, then $\Phi_h(f) = \alpha f + \beta$.

Definition 1. $\Phi_{\mathbf{h}}$ is called a deformation of the space $\mathrm{Re}\mathcal{A} = \{u \in \in C_{\mathbb{R}}(\Omega) : u + iv \in \mathcal{A}\}$, if $\Phi_h(u) \in \mathrm{Re}\mathcal{A}$ for every $u \in \mathrm{Re}\mathcal{A}$.

Each affine mapping $\mathbf{h}(t) = \alpha t + \beta$, where $\alpha, \beta \in \mathbb{R}$, generates a deformation on $\mathrm{Re}\mathcal{A}$ by the formula $\Phi_h(u) = \alpha u + \beta \in \mathrm{Re}\mathcal{A}$.

The family of continuous deformations of the space $\mathrm{Re}\mathcal{A}$ forms a subgroup under the operation of superposition. The family of all linear functions $h(t) = \alpha t + \beta$, where $\alpha, \beta \in \mathbb{R}$, forms a subgroup of affine deformations.

Let $\mathcal{K}(\mathcal{A})$ be the family of continuous deformations of $\mathrm{Re}\mathcal{A}$. Then $\mathcal{K}(\mathcal{A})$ is semigroup with respect to superposition. We denote by $\mathcal{K}_a(\mathcal{A})$ the semigroup of all affine deformations, which is a subgroup of $\mathcal{K}(\mathcal{A})$.

It is easy to see, that if $\mathcal{A} = C_\beta(\Omega)$, then $\mathcal{K}(\mathcal{A}) = C_b(\Omega)$ is a semigroup with respect to superposition .

This note is devoted to the proof of the following assertion.

Theorem 1. Let $\mathcal{A}$ be a β-uniform algebra on a locally compact Hausdorff space Ω. Then $\mathcal{A} = C_\beta(\Omega)$, if and only if there exists at least one non-affine continuous deformation on $\mathrm{Re}\mathcal{A}$.

Proof. If $\mathcal{A} = C_\beta(\Omega)$, then as we have mentioned above, $\mathcal{K}(\mathcal{A}) = C_b(\mathbb{R})$, which means that there is a non-affine continuous deformation on $\mathrm{Re}\mathcal{A}$.

Conversely, let Φ acts on $\mathrm{Re}\mathcal{A}$ as a non-affine continuous deformation, i.e.

$$\Phi : \mathrm{Re}\mathcal{A} \to \mathrm{Re}\mathcal{A}.$$

Let Ω_0 be the Stone–Cech $\mathcal{A}$-compactification of Ω, and $\mathcal{A}_{(0)}$ is the uniform algebra on the compact Ω_0 mentioned above.

Consider a linear extension operator $P : \mathcal{A} \overset{\text{on}}{\rightleftarrows} \mathcal{A}_{(0)}$, such that $Pf = \hat{f}$ and $P^{-1}\hat{f} = \hat{f}|_{\Omega} = f$. By the Banach Theorem on the inverse operator , the operators P and P^{-1} are continuous: $\mathcal{A}_{(0)} \overset{P^{-1}}{\to} \mathcal{A} \overset{\Phi}{\to} \mathcal{A} \overset{P}{\to} \mathcal{A}_{(0)}$, i.e. $\mathrm{Re}\mathcal{A}_{(0)} \overset{P^{-1}}{\to} \mathrm{Re}\mathcal{A} \overset{\Phi}{\to} \mathrm{Re}\,\mathcal{A} \overset{P}{\to}$ $\overset{P}{\to} \mathrm{Re}\mathcal{A}_{(0)}$. Since the operators P, P^{-1} are continuous , and the deformation Φ is continuous and non-affine by assumption, the deformation of the $P \cdot \Phi \cdot P^{-1}$ is also non-affine, continuous deformation on $\mathrm{Re}\mathcal{A}_{(0)}$. But then, by the theorem of [9], we have that $\mathcal{A}_{(0)} = C(\Omega_0)$ and so, $\mathcal{A} = C_\beta(\Omega)$. □

Theorem 2. Let $\mathcal{A}$ be a β-uniform algebra on a locally compact space Ω. Then the discontinuous deformation Φ acts on Re$\mathcal{A}$, if and only if $\mathcal{A}$ is finite .

Proof. Let $\mathcal{A}$ be a β-uniform subalgebra of $C_\beta(\Omega)$, and Φ be a non-discontinuous deformation, which acts on Re$\mathcal{A}$. Consider, as above, the linear extension operator $P : \mathcal{A} \overset{\text{on}}{\rightleftarrows} \mathcal{A}_{(0)}$.

Suppose that $\dim \mathcal{A} = \infty$, then $\dim \mathcal{A}_{(0)} = \infty$. But on the other hand, since the deformation Φ, acting on Re$\mathcal{A}$, is discontinuous, then the deformation of $P \circ \Phi \circ P^{-1}$, which acts on Re$\mathcal{A}_0$, is also discontinuous. Then, by the theorem of [10], we have that $\dim \mathcal{A}_{(0)} < \infty$, and since $\dim \mathcal{A} \leqslant \dim \mathcal{A}_{(0)}$, then $\dim \mathcal{A} < \infty$. □

As a consequence , we note the following results:

Corollary 1. If for a β-uniform algebra $\mathcal{A}$ the deformation of $\Phi_{\mathbf{h}}$, generated by the function $\mathbf{h}(t) = t^2$ acts on Re$\mathcal{A}$, then $\mathcal{A} = C_\beta(\Omega)$.

Corollary 2. If for a β-uniform algebra $\mathcal{A}$ the deformation of $\Phi_{\mathbf{h}}$, generated by the function $\mathbf{h}(t) = |t|$ acts on Re$\mathcal{A}$, then $\mathcal{A} = C_\beta(\Omega)$.

Note that the Corollaries 1 and 2 are improvements for of Wermer [11] and Bernard [12] corresponding theorems in the case of β-uniform algebras.

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