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Mathematics

ON THE REPRESENTATION OF $\langle\rho_j,\ W_j\rangle$ ABSOLUTE MONOTONE FUNCTIONS (part 2)

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In the paper [1] particularly a concept of $\langle\rho_j,\ W_j\rangle$ absolutely monotone function was introduced. In the present paper some representation problems of such functions are investigated. The proofs of some fundamental lemmas and the main theorem are given.

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1. Introduction. Preliminary information was given in the first part of this research [2]. Here we will bring the definitions of fundamental operators and function systems. We denote

$$A_n^* f(x)\equiv\prod_{j=0}^{n} D_j f(x),\ \ n\geq 0,\ D_j f(x)=D^{1/\rho_j}\left\{\frac{f(x)}{W_j(x)}\right\},\ \ j\geq 0,$$

$$\tilde{A}_n^* f(x)=D^{-\alpha_n}\left\{\frac{A_{n-1}^* f(x)}{W_n(x)}\right\},\ \ n\geq 0,\ \ A_{-1}^* f(x)\equiv f,$$

where $\rho_0=1,\ \rho_j\geq 1,\ j\geq 1,\ 1-\alpha_j=\dfrac{1}{\rho_j}, W_0\equiv 1,\ 0<W_j(x)\in C^\infty[0,l),$

$$U_0(x)\equiv 1,\quad U_1(x)=\frac{1}{\Gamma(\rho_1^{-1})}\int_0^x \xi_1^{1/\rho_1-1}W_1(\xi_1)d\xi_1,\ldots,$$

$$U_n(x)=\frac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_0^x W_1(\xi_1)d\xi_1\int_0^{\xi_1}(\xi_1-\xi_2)^{1/\rho_1-1}W_2(\xi_2)d\xi_2\times\cdots\times$$

$$\times\int_0^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{1/\rho_{n-1}-1}\xi_n^{1/\rho_n-1}W_n(\xi_n)d\xi_1,\quad n\geq 2,$$

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$$\Phi_0(t,x)=\begin{cases}1, & 0\le t<x<l,\\ 0, & x\le t<l,\end{cases}$$

$$\Phi_1(t,x)=\begin{cases}\dfrac{1}{\Gamma(\rho_1^{-1})}\displaystyle\int_t^x(\xi_1-t)^{\frac{1}{\rho_1}-1}W_1(\xi_1)d\xi_1, & 0\le t<x<l,\\ 0, & x\le t<l,\end{cases}$$

$$\Phi_n(t,x)=\begin{cases}\dfrac{1}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\displaystyle\int_t^x W_1(\xi_1)d\xi_1\int_t^{\xi_1}(\xi_1-\xi_2)^{\frac{1}{\rho_1}-1}W_2(\xi_2)d\xi_2\times\cdots\times\\ \times\displaystyle\int_t^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}(\xi_n-t)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n, \ 0\le t<x<l,\\ 0, \qquad x\le t<l.\end{cases}$$

2. Some Lemmas.

Lemma 2.1. Let $f(x)\in C_\infty\{[0,l),\langle\rho_j,\ W_j\rangle\}$, $\{W_j(x)\}_0^\infty\in\bar{W}$ and

$$|\tilde{A}_j^*f(t)|\le\frac{C\Gamma(1-\lambda_j)}{(d-t)^{\lambda_j}}\cdot\frac{1}{\prod\limits_{i=1}^{j}m_i(t,d)},\ j\ge1,\ 0\le t<d<l, \tag{2.1}$$

where $\lambda_j=\sum\limits_{k=1}^{j}\dfrac{1}{\rho_k}$, $m_i(t,d)=\min\limits_{x\in[t,d]}W_i(x)$. Then for any x_0 $(0\le x_0<l)$ there exists $\delta>0$ such that for any $x\in[x_0,x_0+\delta]$ we have the equality

$$\lim_{n\to\infty}\int_{x_0}^{x}\Phi_n(t,x)A_n^*f(t)dt=0. \tag{2.2}$$

Proof. Using the definition of the operators A_n^*, $\tilde{A}_n^*$, it is easy to obtain

$$\begin{aligned}\int_{x_0}^{x}\Phi_n(t,x)A_n^*f(t)dt&=\int_{x_0}^{x}\Phi_n(t,x)\frac{d}{dt}\{\tilde{A}_n^*f(t)\}dt=\\ &=\Phi_n(t,x)\tilde{A}_n^*f(t)\Big|_{x_0}^{x}+\int_{x_0}^{x}\tilde{A}_n^*f(t)\frac{d}{dt}\{-\Phi_n(t,x)\}dt=\\ &=-\Phi_n(x_0,x)\tilde{A}_n^*f(x_0)+\int_{x_0}^{x}\tilde{A}_n^*f(t)\frac{d}{dt}\{-\Phi_n(t,x)\}dt.\end{aligned} \tag{2.3}$$

Since $\dfrac{d}{dt}\{-\Phi_n(t,x)\}\ge0$ (Lemma 2.3, [2]) and $\Phi_n(x_0,x)>0$, if $0\le x_0<t<$ $<x<x_0^*<l$, according to (2.1) and (2.3), we obtain

$$\Big|\int_{x_0}^{x}\Phi_n(t,x)A_n^*f(t)dt\Big|\le\Big|\tilde{A}_n^*f(x_0)\Big|\Phi_n(x_0,x)+\int_{x_0}^{x}\Big|\tilde{A}_n^*f(t)\Big|\frac{d}{dt}\{-\Phi_n(t,x)\}dt\le$$

$$\le\frac{C\Gamma(1+\lambda_n)}{(x_0^*-x)^{\lambda_n}}\cdot\frac{\Phi_n(x_0,x)}{\prod\limits_{j=1}^{n}m_j(x_0,x_0^*)}+\int_{x_0}^{x}\frac{C\Gamma(1+\lambda_n)}{(x_0^*-t)^{\lambda_n}\prod\limits_{j=1}^{n}m_j(t,x_0^*)}\frac{d}{dt}\{-\Phi_n(t,x)\}dt\le$$

$$\leq 2\frac{C\Gamma(1+\lambda_n)}{(x_0^*-x)^{\lambda_n}}\cdot\frac{\Phi_n(x_0,x)}{\prod\limits_{j=1}^{n}m_j(x_0,x_0^*)}. \tag{2.4}$$

Using the definition of the function $\Phi_n(x_0,x)$, we can obtain the estimate

$$\Phi_n(x_0,x)\leq\frac{\prod\limits_{j=1}^{n}M_j(x_0,x)}{\Gamma(1+\lambda_n)}(x-x_0)^{\lambda_n}\leq\frac{\prod\limits_{j=1}^{n}M_j(x_0,x_0^*)}{\Gamma(1+\lambda_n)}(x-x_0)^{\lambda_n}. \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$\Big|\int_{x_0}^{x}\Phi_n(t,x)A_n^*f(t)dt\Big|\leq 2C\prod_{j=1}^{n}\frac{M_j(x_0,x_0^*)}{m_j(x_0,x_0^*)}\Big(\frac{x-x_0}{x_0^*-x}\Big)^{\lambda_n}. \tag{2.6}$$

Since $\{W_j(x)\}_0^\infty\in\bar{W}$, the sequence $\{W_j(x)\}_0^\infty$ has the property "ε" [2], consequently, if $\dfrac{x-x_0}{x_0^*-x}<\varepsilon$, then the right side of (2.6) vanishes when $n\to\infty$. It is easy to see that

$$0<\delta<\frac{\varepsilon}{1+\varepsilon}(x_0^*-x_0). \qquad \square$$

L e m m a 2 . 2 . If $f(x)\in C_\infty\{[0,l),\ \langle\rho_j,\ W_j\rangle\}$, $\{W_j(x)\}_0^\infty\in\bar{W}$, and $A_j^*f(x)\geq 0, j\geq 0$, then for any $x_0\in[0,l)$ there exists $\delta>0$ such that the equality

$$\lim_{n\to\infty}\int_{x_0}^{x}\Phi_n(t,x)A_n^*f(t)dt=0 \tag{2.7}$$

holds for any $x\in[x_0,x_0+\delta]$.

P r o o f. In [3] it is shown, that

$$f(x)=\sum_{k=0}^{n}\tilde{A}_k^*f(0)U_k(x)+\int_0^x\Phi_n(t,x)A_n^*f(t)dt. \tag{2.8}$$

From the definition of the operators A_n^*, $\tilde{A}_n^*$ it follows that $\tilde{A}_n^*f(x)\geq 0$ and $\dfrac{d}{dx}\tilde{A}_n^*f(x)=A_n^*f(x)\geq 0$. Taking $x_0^*\in[0,l)$, it is obvious, that

$$f(x_0^*)>\int_0^{x_0^*}\Phi_n(t,x_0^*)A_n^*f(t)dt. \tag{2.9}$$

Using Lemma 3.4 [1] and the formula (2.10), [4], we can get

$$\begin{aligned}
\int_0^{x_0^*}\Phi_n(t,x_0^*)A_n^*f(t)dt &= \int_0^{x_0^*}\Phi_n(t,x_0^*)W_{n+1}(t)\frac{A_n^*f(t)}{W_{n+1}(t)}dt=\\
&=\int_0^{x_0^*}\frac{d}{dt}\Big\{-D_{x_0^*}^{-\alpha_{n+1}}\Phi_{n+1}(t,x_0^*)\Big\}\frac{A_n^*f(t)}{W_{n+1}(t)}dt=\\
&=\int_0^{x_0^*}\Big\{-\frac{d}{dt}\Phi_{n+1}(t,x_0^*)\Big\}{}_0D^{-\alpha_{n+1}}\Big\{\frac{A_n^*f(t)}{W_{n+1}(t)}\Big\}dt=\\
&=\int_0^{x_0^*}\Big\{-\frac{d}{dt}\Phi_{n+1}(t,x_0^*)\Big\}\tilde{A}_{n+1}^*f(t)dt\geq\\
&\geq\int_x^{x_0^*}\Big\{-\frac{d}{dt}\Phi_{n+1}(t,x_0^*)\Big\}\tilde{A}_{n+1}^*f(t)dt\geq
\end{aligned} \tag{2.10}$$

$$\geq\tilde{A}_{n+1}^*f(x)\int_x^{x_0^*}\Big\{-\frac{d}{dt}\Phi_{n+1}(t,x_0^*)\Big\}dt=\tilde{A}_{n+1}^*f(x)\Phi_{n+1}(x,x_0^*).$$

From the definition of the function $\Phi_{n+1}(x,x_0^*)$ we can get the estimation

$$\Phi_{n+1}(x,x_0^*) \geq \prod_{j=1}^{n+1} m_j(x,x_0^*)\cdot\frac{(x_0^*-x)^{\lambda_{n+1}}}{\Gamma(1+\lambda_{n+1})}, \tag{2.11}$$

where $m_j(x,x_0^*) = \min\limits_{t\in[x,x_0^*]} W_j(t)$. From (2.10) and (2.11) it follows that

$$\tilde{A}^*_{n+1}f(x) \leq \frac{f(x_0^*)\Gamma(1+\lambda_{n+1})}{(x_0^*-x)^{\lambda_{n+1}}}\cdot\frac{1}{\prod\limits_{j=1}^{n+1} m_j(x,x_0^*)}. \tag{2.12}$$

Note that (2.12) is similar to the condition (2.1) of Lemma 2.1. □

Lemma 2.3. If $f(x)\in D\big\{[0,l),\langle\rho_j,\ W_j\rangle\big\}$, then for any $k\geq 0,\ n\geq 0$

$$A_k^*\Big(\int_0^x \Phi_n(t,x)A_n^*f(t)dt\Big) \geq 0, \tag{2.13}$$

$$\tilde{A}_k^*\Big(\int_0^x \Phi_n(t,x)A_n^*f(t)dt\Big) \geq 0. \tag{2.14}$$

Proof. Note that from the definition of the operators A_k^* and $\tilde{A}_k^*$ it follows that the inequality (2.14) holds, whenever (2.13) is true. Since $\Phi_n(t,x)\equiv 0$, when $n\geq 0$ and $x\leq t<l<\infty$, we shall prove (2.13) for $0\leq t<x<l$. We denote

$$\psi_{n-k}\Big(t,x,\big\{W_j\big\}_{k+1}^n\Big) \equiv \frac{1}{\prod\limits_{j=k+1}^{n}\Gamma(\rho_j^{-1})}\int_t^x (x-\xi_{k+1})^{\frac{1}{\rho_k}-1}W_{k+1}(\xi_{k+1})d\xi_{k+1}\times$$

$$\times\int_t^{\xi_{k+1}}(\xi_{k+1}-\xi_{k+2})^{\frac{1}{\rho_{k+1}}-1}W_{k+2}(\xi_{k+2})d\xi_{k+2}\times\cdots\times \tag{2.15}$$

$$\times\int_t^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}(\xi_n-t)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n,\quad 0\leq t<x<l,\quad 0\leq k\leq n-2,$$

$$\psi_1\Big(t,x,\big\{W_n\big\}\Big) = \frac{1}{\Gamma(\rho_n^{-1})}\int_t^x (x-\xi_n)^{\frac{1}{\rho_{n-1}}-1}(\xi_n-t)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n. \tag{2.16}$$

Let $k=0$. Using the definition of the operator A_0^*, we have

$$A_0^*\Big(\int_0^x \Phi_n(t,x)A_n^*f(t)dt\Big) = \frac{d}{dx}\Big(\int_0^x \Phi_n(t,x)A_n^*f(t)dt\Big) = \Phi_n(x,x)A_n^*f(x)+$$

$$+\int_0^x \frac{d}{dx}\big\{\Phi_n(t,x)\big\}A_n^*f(t)dt = \int_0^x \frac{d}{dx}\big\{\Phi_n(t,x)\big\}A_n^*f(t)dt =$$

$$= \int_0^x A_n^*f(t)dt\frac{d}{dx}\Big\{\Big(\prod_{j=1}^{n}\Gamma(\rho_j^{-1})\Big)^{-1}\int_t^x W_1(\xi_1)d\xi_1\int_t^{\xi_1}(\xi_1-\xi_2)^{\frac{1}{\rho_1}-1}W_2(\xi_2)d\xi_2\times$$

$$\times\cdots\times\int_t^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}(\xi_n-t)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n\Big\}= \tag{2.17}$$

$$=\frac{W_1(x)}{\prod\limits_{j=1}^{n}\Gamma(\rho_j^{-1})}\int_0^x A_n^*f(t)\Big(\int_t^x(x-\xi_2)^{\frac{1}{\rho_1}-1}W_2(\xi_2)d\xi_2\cdots$$

$$\cdots\int_t^{\xi_{n-1}}(\xi_{n-1}-\xi_n)^{\frac{1}{\rho_{n-1}}-1}(\xi_n-t)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n\Big)dt=$$

$$=\frac{W_1(x)}{\Gamma(\rho_1^{-1})}\int_0^x A_n^*f(t)\psi_{n-1}\Big(t,x,\{W_j\}_2^n\Big)dt\geq 0.$$

Suppose $k=1$. Using the definition of the operator A_1^* and the formula (2.17), changing the integration order and taking into account the equation

$$\int_t^x(x-\tau)^{\alpha_1-1}(\tau-t)^{\frac{1}{\rho_1}-1}d\tau=\Gamma(\alpha_1)\Gamma(\rho_1^{-1}),\ \text{since},\ \Gamma\Big(\alpha_1+\frac{1}{\rho_1}\Big)=1,$$

we get

$$A_1^*\Big(\int_0^x\Phi_n(t,x)A_n^*f(t)dt\Big)=D_1A_0^*\Big(\int_0^x\Phi_n(t,x)A_n^*f(t)dt\Big)=$$

$$=\frac{1}{\Gamma(\rho_1^{-1})}\frac{d}{dx}D^{-\alpha_1}\Big(\int_0^x A_n^*f(t)\psi_{n-1}\Big(t,x,\{W_j\}_2^n\Big)dt\Big)=$$

$$=\frac{1}{\Gamma(\rho_1^{-1})\Gamma(\alpha_1)}\cdot\frac{d}{dx}\int_0^x(x-\tau)^{\alpha_1-1}d\tau\int_0^\tau A_n^*f(t)\psi_{n-1}\Big(t,\tau,\{W_j\}_2^n\Big)dt=$$

$$=\frac{\frac{d}{dx}\int_0^x A_n^*f(t)dt\int_t^x(x-\tau)^{\alpha_1-1}d\tau\int_t^\tau(\tau-\xi_2)^{\frac{1}{\rho_1}-1}W_2(\xi_2)\psi_{n-2}\Big(t,\xi_2,\{W_j\}_3^n\Big)d\xi_2}{\Gamma(\rho_2^{-1})\Gamma(\rho_1^{-1})\Gamma(\alpha_1)}=$$

$$=\frac{\frac{d}{dx}\int_0^x A_n^*f(t)dt\int_t^x W_2(\xi_2)\psi_{n-2}\Big(t,\xi_2,\{W_j\}_3^n\Big)d\xi_2\int_{\xi_1}^x(x-\tau)^{\alpha_1-1}(\tau-\xi_2)^{\frac{1}{\rho_1}-1}d\tau}{\Gamma(\rho_1^{-1})\Gamma(\rho_2^{-1})\Gamma(\alpha_1)}=$$

$$=\frac{1}{\Gamma(\rho_2^{-1})}\cdot\frac{d}{dx}\int_0^x A_n^*f(t)dt\int_t^x W_2(\xi_2)\psi_{n-2}\Big(t,\xi_2,\{W_j\}_3^n\Big)d\xi_2=$$

$$=\frac{1}{\Gamma(\rho_2^{-1})}\cdot\frac{d}{dx}\int_0^x W_2(\xi_2)d\xi_2\int_0^{\xi_2}A_n^*f(t)\psi_{n-2}\Big(t,\xi_2,\{W_j\}_3^n\Big)dt=$$

$$=\frac{W_2(x)}{\Gamma(\rho_2^{-1})}\int_0^x A_n^*f(t)\psi_{n-2}\Big(t,x,\{W_j\}_3^n\Big)dt\geq 0. \tag{2.18}$$

By the same argument, in the case $0\leq k\leq n-2$ we have

$$A_k^*\Big(\int_0^x\Phi_n(t,x)A_n^*f(t)dt\Big)=\frac{W_{k+1}(x)}{\Gamma(\rho_{k+1}^{-1})}\int_0^x A_n^*f(t)\psi_{n-k-1}\Big(t,x,\{W_j\}_{k+2}^n\Big)dt\geq 0.$$

Now suppose $k=n-1$. Using the definition of the operator D_{n-1}, the formulas (2.16) (for $k=n-2$) and (1.19), we obtain

$$A^*_{n-1}\Big\{\int_0^x \Phi_n(t,x)A^*_n f(t)dt\Big\} = D_{n-1}A^*_{n-2}\Big(\int_0^x \Phi_n(t,x)A^*_n f(t)dt\Big) =$$

$$= \frac{1}{\Gamma(\rho^{-1}_{n-1})}\frac{d}{dx}D^{-\alpha_{n-1}}\Big(\int_0^x A^*_n f(t)\psi_1\Big(t,x,\{W_n\}\Big)dt\Big) =$$

$$= \frac{1}{\Gamma(\rho^{-1}_{n-1})\Gamma(\alpha_{n-1})}\cdot\frac{d}{dx}\int_0^x (x-\tau)^{\alpha_{n-1}-1}d\tau\int_0^\tau A^*_n f(t)\psi_1\big(t,\tau,\{W_n\}\big)dt =$$

$$= \frac{\dfrac{d}{dx}\displaystyle\int_0^x A^*_n f(t)dt\int_t^x (x-\tau)^{\alpha_{n-1}-1}d\tau\int_0^\tau (\tau-\xi_n)^{\frac{1}{\rho_{n-1}}-1}(\xi_n-t)^{\frac{1}{\rho_n}-1}W_n(\xi_n)d\xi_n}{\Gamma(\rho_n^{-1})\Gamma(\rho^{-1}_{n-1})\Gamma(\alpha_{n-1})} =$$

$$= \frac{1}{\Gamma(\rho_n^{-1})}\cdot\frac{d}{dx}\int_0^x W_n(\xi_n)d\xi_n\int_0^{\xi_n}(\xi_n-t)^{\frac{1}{\rho_n}-1}A^*_n f(t)dt =$$

$$= \frac{1}{\Gamma(\rho_n^{-1})}W_n(x)\int_0^x (x-t)^{\frac{1}{\rho_n}-1}A^*_n f(t)dt \geq 0. \qquad (2.19)$$

Let $k=n$. Then, using the definition of the operator A^*_n and the formula (2.19), we get

$$A^*_n\Big\{\int_0^x \Phi_n(t,x)A^*_n f(t)dt\Big\} = D_n A^*_{n-1}\Big(\int_0^x \Phi_n(t,x)A^*_n f(t)dt\Big) = \cdots$$

$$\cdots = \frac{1}{\Gamma(\rho_n^{-1})\Gamma(\alpha_n)}\cdot\frac{d}{dx}\int_0^x A^*_n f(t)dt\int_t^x (x-\tau)^{\alpha_n-1)}(\tau-t)^{\frac{1}{\rho_n}-1}d\tau = \qquad (2.20)$$

$$= \frac{d}{dx}\int_0^x A^*_n f(t)dt = A^*_n f(x) \geq 0,$$

and finally

$$A^*_{n+1}\Big\{\int_0^x \Phi_n(t,x)A^*_n f(t)dt\Big\} = D_{n+1}A^*_n\Big(\int_0^x \Phi_n(t,x)A^*_n f(t)dt\Big) = D_{n+1}A^*_n f(x) =$$

$$= \frac{d}{dx}D^{-\alpha_{n+1}}\Big\{\frac{A^*_n f(x)}{W_{n+1}(x)}\Big\} = A^*_{n+1}f(x) \geq 0. \qquad \square$$

3. Main Theorem.

Theorem. If $f(x)\in D\big\{[0,l),\ \langle\rho_j,\ W_j\rangle\big\}$, where $\big\{W_j\big\}_0^\infty \in \bar{W}$, then

$$f(x) = \sum_{k=0}^\infty \tilde{A}^*_k f(0)U_k(x), \quad x\in[0,l). \qquad (3.1)$$

Proof. It is known [1], that

$$f(x) = \sum_{k=0}^n \tilde{A}^*_k f(0)U_k(x) + \int_0^x \Phi_n(t,x)A^*_n f(t)dt. \qquad (3.2)$$

We note that $\sum_{k=0}^n \tilde{A}^*_k f(0)U_k(x) \leq f(x)$, and consequently, the series $\sum_{k=0}^\infty \tilde{A}^*_k f(0)U_k(x)$ converges for any $x\in[0,l)$, moreover in each interval $[0,d]$, $0<d<l<+\infty$, the convergence is uniformly, since $U_k(x)\leq U_k(d)$, $x\in[0,d]$. Consider the sequence

$$R_n(x)=\int_0^x \Phi_n(t,x)A_n^* f(t)dt.$$

Obviously $R_n(x)\geq 0,\ \forall x\in[0,l),\ R_{n+1}(x)\leq R_n(x),\ n=1,2,...,$ and consequently $\exists \lim\limits_{n\to\infty} R_n(x)=R(x)$. We denote

$$g(x)=\sum_{k=0}^{\infty}\tilde{A}_k^* f(0)U_k(x).$$

Note, that

$$A_j^* f(x)=\sum_{k=j+1}^{n}\tilde{A}_k^* f(0)A_j^*\Big\{U_k(x)\Big\}+A_j^*\Big\{\int_0^x \Phi_n(t,x)A_n^* f(t)dt\Big\}. \quad (3.3)$$

Since $A_j^*\{U_k(x)\}\geq 0$, $A_j^*\Big\{\int_0^x \Phi_n(t,x)A_n^* f(t)dt\Big\}\geq 0,\ \ j,k,n=0,1,...,$ then

$$\sum_{k=j+1}^{n}\tilde{A}_k^* f(0)A_j^*\{U_k(x)\}\leq A_j^* f(x), \qquad j=0,1,... \quad (3.4)$$

From (3.4) it follows that the series $\sum\limits_{k=j+1}^{\infty}\tilde{A}_k^* f(0)A_j^*\{U_k(x)\}$ converges in $[0,l)$ and, moreover, it converges uniformly in each interval $[0,l_2]$ (where $l_2<l$), since from Lemma 2.4 [2] we have

$$\sum_{k=j}^{\infty}\tilde{A}_k^* f(0)A_j^*\{U_k(x)\}\leq \frac{1}{\Phi_j(x,l_2)}\sum_{k=j}^{\infty}\tilde{A}_k^* f(0)U_k(l_2)\leq \frac{1}{\Phi_j(l_1,l_2)}\sum_{k=j}^{\infty}\tilde{A}_k^* f(0)U_k(l_2), \quad (3.5)$$

when $0<x<l_1<l_2<l$.

It is obvious, that if

$$A_j^* g(x)=\sum_{k=j+1}^{\infty}\tilde{A}_k^* f(0)A_j^*\{U_k(x)\}, \quad (3.6)$$

then

$$\tilde{A}_j^* g(x)=\sum_{k=j+1}^{\infty}\tilde{A}_k^* f(0)\tilde{A}_j^*\{U_k(x)\}, \quad (3.7)$$

since $\tilde{A}_j^* g(x)=D^{-\alpha_j}\Big\{\frac{A_{j-1}^* g(x)}{W_j(x)}\Big\}$. Now let us prove the equality (3.6). Let $j=0$ and, so, $A_0^* g(x)=g'(x)$. We denote

$$\sum_{k=1}^{\infty}\tilde{A}_k^* f(0)U_k'(x)\equiv \varphi_0(x). \quad (3.8)$$

Note that the terms of the series (3.8) are positive and continuous and, thus, by Levi theorem, we have $\sum\limits_{k=1}^{\infty}\tilde{A}_k^* f(0)\int_a^x U_k'(t)dt\equiv\int_a^x \varphi_0(t)dt,\ \ 0<a<x<l,$ that is $g(x)-g(a)=\int_a^x \varphi_0(t)dt$. Hence, we get $g'(x)=A_0^* g(x)=\sum\limits_{k=1}^{\infty}\tilde{A}_k^* f(0)U_k'(x)$. We suppose that (3.6) is true for $j>0$. Let prove the same equality for $j+1$, that is

$$A^*_{j+1} g(x) = \sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) A^*_{j+1}\{U_k(x)\}. \tag{3.9}$$

Using the definition of the operator $\tilde{A}^*_{j+1}$ and the equation (3.6), we can easily obtain

$$\tilde{A}^*_{j+1} g(x) = \sum_{k=j+1}^{\infty} \tilde{A}^*_k f(0) \tilde{A}^*_{j+1}\{U_k(x)\} = \sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) \tilde{A}^*_{j+1}\{U_k(x)\}, \tag{3.10}$$

since $\tilde{A}^*_{j+1}\{U_{j+1}(x)\} = 0$. We denote

$$\sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) A^*_{j+1}\{U_k(x)\} \equiv \varphi_{j+1}(x),\ x \in [0,l).$$

Note that

$$\sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) \int_a^x \frac{d}{dt} \tilde{A}^*_{j+1}\{U_k(t)\} dt = \int_a^x \varphi_{j+1}(t) dt$$

and, so, we get

$$\sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) \tilde{A}^*_{j+1}\{U_k(x)\} - \sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) \tilde{A}^*_{j+1}\{U_k(a)\} = \tilde{A}^*_{j+1} g(x) -$$

$$-\tilde{A}^*_{j+1} g(a) = \int_a^x \varphi_{j+1}(t) dt.$$

Now it is easy to obtain $\frac{d}{dx}\tilde{A}^*_{j+1} g(x) = \varphi_{j+1}(x)$, namely $A^*_{j+1} g(x) =$ $= \sum_{k=j+2}^{\infty} \tilde{A}^*_k f(0) A^*_{j+1}\{U_k(x)\}$. Hence, the equation (3.6) is satisfied for any $j \geq 0$. It is obvious that $g(x) \in C_\infty\{[0,l), \langle \rho_j,\, W_j \rangle\}$, and consequently

$$R(x) = f(x) - g(x) \in C_\infty\{[0,l), \langle \rho_j,\, W_j \rangle\}.$$

According to (2.12) and (3.5), we have

$$\left|\tilde{A}^*_j R(x)\right| \leq \tilde{A}^*_j f(x) + \left|\tilde{A}^*_j g(x)\right| \leq \tilde{A}^*_j f(x) + \frac{1}{\Phi_j(x,l_2)} \sum_{k=1}^{\infty} \tilde{A}^*_k f(0) U_k(l_2) \leq$$

$$\leq \frac{f(l_2)\Gamma(1+\lambda_j)}{(l_2-x)^{\lambda_j}} \cdot \Big(\prod_{i=1}^{j} m_i(x,l_2)\Big)^{-1} + \frac{f(l_2)}{\Phi_j(x,l_2)}, \qquad 0 \leq x < l_2 < l. \tag{3.12}$$

Transforming the estimation (2.11), we get

$$\Phi_j(x,l_2) > \frac{(l_2-x)^{\lambda_j}}{\Gamma(1+\lambda_j)} \cdot \prod_{i=1}^{j} m_i(x,l_2), \qquad 0 \leq x < l_2 < l, \tag{3.13}$$

$$\left|\tilde{A}^*_j R(x)\right| \leq \frac{2f(l_2)\Gamma(1+\lambda_j)}{(l_2-x)^{\lambda_j}} \cdot \Big(\prod_{i=1}^{j} m_i(x,l_2)\Big)^{-1}. \tag{3.14}$$

The function $R(x)$ satisfies the conditions of Lemma 2.1. Let us show that $R(x) \equiv 0,\ x \in [0,l)$. According to Lemma 2.1, there exists δ such that for any $x \in [0,\delta]$ we have $\lim\limits_{n\to\infty} \int_0^x \Phi_n(t,x) A_n^* R(t) dt = 0$, that is $R(x) \equiv 0,\ x \in [0,\delta]$.

Denote $x_0^* = \sup\{\delta\}$, where $[0,x_0^*]$ is the maximal interval when $R(x) = 0$. Note that $\tilde{A}_j^* R(0) = 0,\ j = 0,1,\ldots$ By Lemma 2.1 and Eq. (2.2), there exists $\delta_1 > 0$ corresponding to x_0^* such that for any $x \in [x_0^*, x_0^* + \delta_1]$ we have $\lim\limits_{n\to\infty} \int_{x_0^*}^x \Phi_n(t,x) A_n^* R(t) dt = 0$, it means that $R(x) = 0$ on the interval $[x_0^*, x_0^* + \delta_1]$ and consequently $[0,x_0^*]$ is not the maximal interval when $R(x) = 0$.

We get a contradiction, which shows that $R(x) \equiv 0$ for all $x \in [0,l)$ and, so,

$$f(x) = \sum_{k=0}^{\infty} \tilde{A}_k^* f(0) U_k(x), \qquad x \in [0,l). \qquad \square$$

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R E F E R E N C E S

1. **Sahakyan B.A.** Differential Operators of the Fractional Order and Associeted with Them $\langle \rho_j,\ W_j \rangle$ Absolutely Monotone Functions. // Uchenie Zapiski EGU, 1975, № 1, p. 3–9 (in Russian).
2. **Sahakyan B.A.** On the Representation of $\langle \rho_j,\ W_j \rangle$ Absolute Monotone Functions. // Proceedings of the YSU. Phys. Math. Scienses, 2014, № 1, p. 26–34.
3. **Sahakyan B.A.** PhD Thesis. Yer.: YSU, 1975, p. 1122 (in Russian).
4. **Dzhrbashyan M.M.** Extention of Quasianalytical Classes of Denjoy–Carleman. // Izv. AN Arm. SSR. Ser. Matem., 1968, v. 3, № 34, p. 171–248 (in Russian).