

AN ALTERNATIVE MODEL FOR BONUS-MALUS SYSTEM

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The paper introduces an alternative approach to the bonus-malus system construction. In the presented model the premium calculation is based on the previous premium and on the claim severity component as well. Following to the concept of an optimal bonus-malus system the necessary and sufficient condition for the aggregate premiums to be a martingale series have been found out. Thus the presented approach differs totally from the usual bonus-malus classes as well as from the systems, where the severity of the claim is omitted.

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Introduction. Bonus-Malus system (BMS) is a premium calculation system, which penalizes the policyholders responsible for one or more claims by a premium surcharge (malus) and rewards the policyholders, who had a claim free year by awarding discount of the premium (bonus).

The majority of optimal BMS presented up to now in the actuarial literature [1] assign to each policyholder a premium based on the number of his accidents. In this way a policyholder, who had an accident with a small size of loss is penalized unfairly in the same way with a policyholder, who had an accident with a big size of loss.

From practical point of view it is well known several considerable disadvantages of existing BMS possess, which are difficult or even impossible to handle within the traditional theory of experience rating [2]. In particular, the existing systems are based on the following characteristic: the claim amounts are omitted as a posterior tariff criterion. This characteristic leads to the following disadvantages.

i. Regarding an occurred claim the future loss of bonus will in many cases exceed the claim amount.

ii. In many cases it gives the policyholder a feeling of unfairness, especially when a policyholder make a small claim and the other one a large, they have the same penalty within the same risk group.

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iii. Consequence of (i and ii) is the well-known bonus hunger behavior of policyholders.

iv. Bonus hunger behavior leads to an asymmetric information between the policyholders, insurers and regulators.

Many authors have focused on the disadvantages mentioned above, in particular the problem of bonus hunger [3,4], the problem of asymmetric information [5,6]. The aim of this paper is to introduce an alternative bonus-malus approach, which at least theoretically eliminates the most important ones of these disadvantages.

An Optimal Bonus-Malus Premium Construction Considerations.

Concept of an optimal BMS [3] has been used in this paper.

A BMS is called optimal if it is: a) financially balanced for the insurer, that is the average total amount of bonuses is equal to the average total amount of maluses; b) fair for the policyholder, that is each policyholder pays a premium proportional to the risk that he imposes to the pool.

Optimal BMS can be divided in two categories: those based only on a posteriori classification criteria and those based both on a priori and a posteriori classification criteria. The majority of BMS designed is based on the number of accidents disregarding their severity. Thus, let us consider the design of an optimal BMS based on a claim severity component.

Notations and Definitions. Based on the probabilistic approach and following axioms of probability theory suppose that all the observations made on a probability space (Ω, \mathcal{F}, P) , where Ω is the set of elementary outcomes ω , \mathcal{F} is a σ -algebra of subsets of Ω and P is a given probability measure on \mathcal{F} .

Time and dynamics have a significant role for the model construction, so suppose that a sequence of σ -algebra $\{\mathcal{F}_n\}_{n \geq 0}$ is given:

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_n \dots \subseteq \mathcal{F}.$$

So the basic probabilistic model is $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$ filtered probability space.

Definition 1. Let X_0, X_1, \dots be a series of random variables given on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, P)$. If X_n is \mathcal{F}_n -measurable for any $n \geq 0$, then we will say that $X = (X_n, \mathcal{F}_n)_{n \geq 0}$ collection or just $X = (X_n, \mathcal{F}_n)$ is a stochastic series.

Definition 2. If for $X = (X_n, \mathcal{F}_n)$ stochastic series X_n is \mathcal{F}_{n-1} -measurable as well, it will be written as $X = (X_n, \mathcal{F}_{n-1})$ assuming $\mathcal{F}_{-1} = \mathcal{F}_0$ and X will be called *predictable series*.

Definition 3. Let $X_n : \Omega \rightarrow \mathbb{R}$, then $X = (X_n, \mathcal{F}_n)$ stochastic series will be called *martingale*, if for any $n \geq 0$:

- a) $E |X_n| < \infty$;
- b) $E (X_{n+1} | \mathcal{F}_n) = X_n$.

From properties of conditional expectation it is obvious that the second property of martingale definition can be rephrased by the following:

$$\int_A X_{n+1} dP = \int_A X_n dP$$

for all $n \geq 0$, $A \in \mathcal{F}_n$ and specially if $A = \Omega$, then it can be written that

$$\text{b*) [7] } EX_n = EX_{n-1} = \dots = EX_1 = EX_0.$$

Aggregate Premium as a Martingale. Let us consider a portfolio of an insurance product. Suppose that a series of independent random variables Y_1, Y_2, \dots are yearly aggregate claim losses of that portfolio, given on a $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbf{P})$ filtered probability space, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{Y_1, Y_2, \dots, Y_n\}$. Suppose that Y_1, Y_2, \dots random variables are so that $EY_n < \infty$ condition is satisfies for any $Y_n \geq 0$ and for all $n \geq 1$.

Let denote P_0, P_1, \dots the random variables, which describe yearly aggregate premium charge for that portfolio, where $P_0 = \text{const}$ is given and the other members of that series are defined by the following formulae:

$$P_n = (1 - \alpha_n)P_{n-1} + \beta_n Y_n, \quad (1)$$

where P_n is an aggregate premium collected for n -th year.

Y_n is an aggregate claim loss for the given portfolio within $(n-1; n)$ time interval. It is necessary to note that Y_n is independent of P_{n-1} for all $n \geq 1$.

$\alpha = (\alpha_n, \mathcal{F}_{n-1})_{n \geq 1}$ is a predictable series, which will be called *a series of bonus factors*.

$\beta = (\beta_n, \mathcal{F}_{n-1})_{n \geq 1}$ is also a predictable series, which will be called *a series of malus factors*.

L e m m a . The series $P = (P_n, \mathcal{F}_n)$ constructed by the formulae (1) is a martingale if and only if:

$$\alpha_n P_{n-1} = \beta_n EY_n \quad (2)$$

P r o o f .

Necessity. Let series α_n and β_n be \mathcal{F}_{n-1} -measurable and the series constructed by (1) be a martingale.

Let calculate $E(P_n | \mathcal{F}_{n-1})$ using \mathcal{F}_{n-1} -measurability of α_n and β_n series, independence of Y_n 's, properties of conditional expectation and the definition of martingale:

$$E(P_n | \mathcal{F}_{n-1}) = (1 - \alpha_n)P_{n-1} + \beta_n EY_n.$$

For the (b) condition of martingale definition it is necessary that:

$$\beta_n = \frac{\alpha_n P_{n-1}}{EY_n}.$$

It is obvious that this result is equivalent to (2).

Sufficiency. Let α_n and β_n be \mathcal{F}_{n-1} -measurable series and the relationship (2) holds. Let construct a series of P_n according to (1).

Substituting (2) in (1) and making some rearrangements, we get:

$$P_n = P_{n-1} + \beta_n (Y_n - EY_n). \quad (3)$$

It is easy to see that $EP_n = EP_0 < \infty$.

As Y_n is independent of \mathcal{F}_{n-1} , then

$$E(P_n | \mathcal{F}_{n-1}) = P_{n-1} + \beta_n(E(Y_n | \mathcal{F}_{n-1}) - EY_n) = 0. \quad \square$$

Offering an insurance product, the insurance company wishes to have a financially stable model. For that purpose it states its strategy for that risk portfolio and defines a premium level. The aggregate premium received for that portfolio must be sufficient to cover some level of aggregate claim with appropriate probability, which is defined in the company's strategy. This means that the company states some Y_c critical value of aggregate claim and some ε probability and defines the aggregate premium P_n , so that it is greater than Y_c critical value with $1 - \varepsilon$ probability. It mathematically expresses as:

$$P(P_n > Y_c) = 1 - \varepsilon. \quad (4)$$

So Y_c is a ε order quantile of random variable Y_n .

Finding α_n and β_n presented in (1) is our main purpose, but we can not do it with the help of Lemma only, which gives their relationship (2).

Suppose that the distribution function of aggregate claim is given $Y_n \sim F_{Y_n}(x)$. Let's find α_n and β_n with the help of expressions (2), (4) and financially stable and optimal BMS concepts.

Reforming (4) and taking into consideration (1) we get:

$$F_{Y_n}\left(\frac{Y_c - (1 - \alpha_n)P_{n-1}}{\beta_n}\right) = \varepsilon.$$

Or using inverse distribution function:

$$\frac{Y_c - (1 - \alpha_n)P_{n-1}}{\beta_n} = F_{Y_n}^{-1}(\varepsilon),$$

where

$$F_{Y_n}^{-1}(\varepsilon) = \inf\{x \in R, F_{Y_n}(x) \geq \varepsilon\}.$$

Substituting (2) and making some rearrangements, we get:

$$\beta_n = \frac{Y_c - P_{n-1}}{F_{Y_n}^{-1}(\varepsilon) - EY_n} \quad (5)$$

and

$$\alpha_n = \frac{Y_c - P_{n-1}}{F_{Y_n}^{-1}(\varepsilon) - EY_n} \cdot \frac{EY_n}{P_{n-1}}. \quad (6)$$

Some Examples.

Example 1. Suppose that the yearly aggregate claims of an insurance company are distributed exponentially with λ rate: $Y \sim Exp(\lambda)$. Let's find α_n and β_n for this case.

Using the characteristics of an exponential distribution function we get:

$$\beta_n = \frac{\lambda (P_{n-1} - Y_c)}{1 + \ln(1 - \varepsilon)},$$

$$\alpha_n = \frac{P_{n-1} - Y_c}{P_{n-1}(1 + \ln(1 - \varepsilon))}.$$

Example 2. Suppose that the yearly aggregate claims of an insurance company have a Pareto distribution with parameters μ and λ ($Y \sim \text{Pareto}(\mu, \lambda)$). For finding α_n and β_n we need the inverse of that distribution function and the expectation. So,

$$\beta_n = \frac{Y_c - P_{n-1}}{\lambda(1 - \varepsilon)^{-\frac{1}{\mu}} - \frac{\lambda\mu}{\mu-1}},$$

$$\alpha_n = \frac{Y_c - P_{n-1}}{P_{n-1} \left(\lambda(\mu - 1)(1 - \varepsilon)^{-\frac{1}{\mu}} - \lambda\mu \right)}.$$

Conclusion. Formulae (1) presented in this paper have theoretical and practical significance as well. It can be used as a model for premium calculation in bonus-malus system which eliminates (i–iv) disadvantages of many BMS's. The model constructed according to the second condition of BMS is optimal. On the other hand, the series of bonus and malus factors are calculated according to the first condition of BMS optimality using the concept of martingale.

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