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Mathematics

ON AUTOMORPHISMS OF SOME PERIODIC PRODUCTS OF GROUPS

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It is proved, that if the order of a splitting automorphism of n-periodic product of cyclic groups of order r is a power of some prime, then this automorphism is inner, where $n \geq 1003$ is odd and r divides n. This is a generalization of the analogue result for free periodic groups.

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Introduction. An automorphism φ of a group G is said to be a *splitting automorphism of period n*, if the relation $(\varphi g)^n = 1$ holds for every $g \in G$ in the semidirect product $G \rtimes \langle\varphi\rangle$, i.e. $\varphi^n = 1$ and $g\, g^{\varphi} g^{\varphi^2} \cdots g^{\varphi^{n-1}} = 1$ for all $g \in G$. Many authors studied groups with splitting automorphisms. For example, it is true that every finite or solvable group with a nontrivial splitting automorphism of prime period is a nilpotent group (see [1,2]), and a finite group with splitting automorphism of period 4 is solvable (see [3]). It is true as well, that the splitting automorphism of odd period $n \geq 1003$ is an inner automorphism, if its order is a prime power [4].

Obviously, every inner automorphism of a group, in which the identity $x^n = 1$ holds is a splitting automorphism of period n. However, the converse fails to hold (see Example 1 in [5]). The main result of this paper is the following theorem.

Theorem 1. Let φ be a splitting automorphism of period n of the n-periodic product $F = \prod_{i\in I}^{n} \langle a_i\rangle$ of cyclic groups $\langle a_i\rangle$ of odd order $r \geq 1003$, where r divides n. Then φ is an inner automorphism, if the order of the automorphism φ is a power of some prime.

Some Auxiliary Lemmas. Let F be an n-periodic product of $m > 1$ cyclic groups of an odd order r, where r divides n (for the definition see [6]). In the paper [7] it was constructed a set $\mathcal{M}_n$ of normal subgroups of the group F with the following property: if $N \in \mathcal{M}_n$, then every two non-commuting elements of the quotient group

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F/N have period n and they generate the whole (infinite) group F/N. In particular, it follows from here that F/N is a non-abelian simple group and its center is trivial.

An automorphism φ of a group G is said to be a *normal automorphism*, if $H^{\varphi} = H$ for all normal subgroups $H \lhd G$. The next two statements were proved in the papers [7] and [5] respectively.

Lemma 1. ([7]). Let $F = \prod_{i\in I}^{n} \langle a_i \rangle$ be an n-periodic product of cyclic groups $\langle a_i \rangle$ of an odd order $r \geq 1003$, where r divides n. Then each normal automorphism φ of the group G is inner.

Lemma 2. ([5], Lemma 4). Let φ be an arbitrary automorphism and N be a normal subgroup of a group G such that the quotient group F/N is a non-abelian simple group. Then, if the subgroups $N, N^{\varphi}, \ldots, N^{\varphi^{k-1}}$ are pairwise distinct and $N^{\varphi^k} = N$, then the quotient group $G/\bigcap_{i=1}^{k} N^{\varphi^i}$ is decomposed into the direct product of normal subgroups $N_j/\bigcap_{i=1}^{k} N^{\varphi^i}$, $j = 1, 2, \ldots, k$, where $N_j = \bigcap_{\substack{i=1\\ i\neq j}}^{k} N^{\varphi^i}$, and every quotient group $N_j/\bigcap_{i=1}^{k} N^{\varphi^i}$ is isomorphic to the group G/N.

The analogue of the next lemma for the free periodic groups of period n is proved in [5].

Lemma 3. ([5], Lemma 3). If φ is an arbitrary nontrivial splitting automorphism of a period n of the group $F = \prod_{i\in I}^{n} \langle a_i \rangle$, which is an n-periodic product of cyclic groups $\langle a_i \rangle$ of an odd order $r \geq 1003$, where r divides n for an odd $n \geq 1003$, then the stabilizer of every normal subgroup $N \in \mathcal{M}_n$ with respect to the action of the cyclic group $\langle \varphi \rangle$ is nontrivial.

Proof. Let the order of a splitting automorphism φ be $r > 1$, and let the stabilizer of some subgroup $N \in \mathcal{M}_n$ with respect to the action of the cyclic group $\langle \varphi \rangle$ be nontrivial. We will show that the last assumption leads to a contradiction.

Consider the normal subgroup $K = \bigcap_{i=1}^{r} N^{\varphi^i}$. By Lemma 2, the quotient group F/K is decomposed into the direct product of non-abelian simple subgroups $N_1/K, \ldots, N_r/K$, where $N_j = \bigcap_{\substack{i=1\\ i\neq j}}^{r} N^{\varphi^i}$, $j = 1, 2, \ldots, r$. It is easy to check that $N_j^{\varphi^t} = N_s$, where $t + j \equiv s (\operatorname{mod} r)$.

By the same Lemma, the quotient group N_r/K is isomorphic to the group F/N, where $N \in \mathcal{M}_n$. Therefore, there exists an element $x \in F$ such that $x \in N_r = \bigcap_{i=1}^{r-1} N^{\varphi^i}$, $x \notin K = \bigcap_{i=1}^{r} N^{\varphi^i} = N_r \cap N$ and xK has order n in the group N_r/K. Since $x \notin K$, then $x \notin N$. Furthermore, by the condition $x \in N_r \setminus K$, we have the relations $x^{\varphi^t} \in N_t$ and $x^{\varphi^t} \notin K$, $t = 1, 2, \ldots, r$. By the conditions of Theorem, we have

$$x \cdot x^{\varphi} \cdot x^{\varphi^2} \cdot \ldots \cdot x^{\varphi^{n-1}} = 1. \tag{1}$$

Since $\varphi^r = 1$, we get

$$x \cdot x^{\varphi} \cdot x^{\varphi^2} \cdot \ldots \cdot x^{\varphi^{n-1}} = \left(x \cdot x^{\varphi} \cdot x^{\varphi^2} \cdot \ldots \cdot x^{\varphi^{r-1}}\right)^{\frac{n}{r}}. \tag{2}$$

Then equation $(xKx^{\varphi}Kx^{\varphi^2}K\dots x^{\varphi^{r-1}}K)^{n/r} = (xK)^{n/r}(x^{\varphi}K)^{n/r}(x^{\varphi^2}K)^{n/r}\dots(x^{\varphi^{r-1}}K)^{n/r}$ holds in quotient group F/K.

Thus, by Eqs. (1), (2),

$$x \cdot x^{\varphi} \cdot x^{\varphi^2} \cdot \ldots \cdot x^{\varphi^{n-1}}K = (x^{\frac{n}{r}})K(x^{\frac{n}{r}})^{\varphi}K(x^{\frac{n}{r}})^{\varphi^2}K\dots(x^{\frac{n}{r}})^{\varphi^{r-1}}K = K. \quad (3)$$

The multiplier of the product

$$(x^{\frac{n}{r}})K(x^{\frac{n}{r}})^{\varphi}K(x^{\frac{n}{r}})^{\varphi^2}K\dots(x^{\frac{n}{r}})^{\varphi^{r-1}}K$$

belong to distinct direct components of the group F/K. Thus, by Eq. (3)

$$(x^{\frac{n}{r}})K = K, \quad (x^{\frac{n}{r}})^{\varphi}K = K, \quad \dots, \quad (x^{\frac{n}{r}})^{\varphi^{r-1}}K = K.$$

In particular, we have $(x^{\frac{n}{r}})K = K$, where $r > 1$. But it is a contradiction since the element xK is of order n in the group N_r/K. The contradiction proves the Lemma. □

The Proof of the Main Result. By Lemma 1, it suffices to show that for every normal subgroup $N \in \mathcal{M}_n$ the equation $N^{\phi} = N$ holds.

We carry out the proof by contradiction. Denote $G = F = \prod_{i\in I}^{n}\langle a_i\rangle$ and suppose that there are normal subgroups $A \in \mathcal{M}_n$ that are not ϕ-invariant subgroups, i.e. such that $A^{\phi} \neq A$. On the other hand, by Lemma 3, the stabilizer of every subgroup A of this kind is nontrivial. Let p^r be the order of the automorphism ϕ, where p is a prime. Obviously, p^r divides n. Since the subgroups of cyclic groups of order p^r are totally ordered with respect to inclusion, then among all non-ϕ-invariant subgroups $A \in \mathcal{M}_n$ one can choose a subgroup with a minimal stabilizer, which we denote by N. This minimal nontrivial stabilizer, regarded as a subgroup of the group $\langle\phi\rangle$ of order p^r, is generated by some automorphism of the form ϕ^{p^k}, where $1 < k < r$. Since the subgroup $\langle\phi^{p^k}\rangle$ is minimal, it is contained in the stabilizer of every subgroup $A \in \mathcal{M}_n$. Therefore, the automorphism ϕ^{p^k} stabilizes all subgroups $A \in \mathcal{M}_n$. By Lemma 1, this implies that the automorphism ϕ^{p^k} is inner.

We have $N^{\phi} \neq N$. Since $N \in \mathcal{M}_n$, it follows that the quotient group F/N is a non-abelian simple group. Applying Lemma 2 to the group $G = F$, we obtain that the quotient group F/K is decomposed into the direct product of the subgroups N_0/K, N_1/K, ... , N_{p^k-1}/K, where $K = \bigcap_{i=0}^{p^k-1} N^{\phi^i}$.

Since the automorphism ϕ^{p^k} is inner, then $\phi^{p^k} = i_u$ for some element $u \in F$. Since the automorphism ϕ^{p^k} is of order p^{r-k}, then the element $u^{p^{r-k}}$ belongs to the center of the group F. By triviality of the center of F it follows $u^{p^{r-k}} = 1$.

For the element uK of the quotient group F/K there are uniquely defined elements $u_0K, u_1K, ..., u_{p^k-1}K$, belonging to the subgroups N_0/K, N_1/K, ... , N_{p^k-1}/K respectively such that

$$uK = u_0K \cdot u_1K \cdot \ldots \cdot u_{p^k-1}K. \quad (4)$$

We can choose element $a \in F$ such that $aK \in N_0/K$ and the element au_0K is in order n. By Eq. (4) the relations

$$u^s a u^{-s} K = u_0^s a u_0^{-s} K \quad (5)$$

hold for every integer s.

Since ϕ is a splitting automorphism, it follows that $a a^{\phi} a^{\phi^2} \cdots a^{\phi^{n-1}} = 1$. Hence

$$aK \cdot a^{\phi} K \cdot a^{\phi^2} K \cdot \ldots \cdot a^{\phi^{n-1}} K = K. \tag{6}$$

Represent the Eq. (6) in the form

$$bK \cdot b^{\phi} K \cdot b^{\phi^2} K \cdot \ldots \cdot b^{\phi^{p^k-1}} K = K, \tag{7}$$

where

$$b = a \cdot a^{\phi^{p^k}} \cdot a^{\phi^{2p^k}} \cdot \ldots \cdot a^{\phi^{(n/p^k-1)p^k}}. \tag{8}$$

It follows from (7) that all the factors on the left-hand side of the equation are trivial. Since $\phi^{p^k} = i_u$, one can write Eq. (8) in the form

$$b = a \cdot uau^{-1} \cdot u^2 au^{-2} \cdot u^{n/p^k-1} au^{-(n/p^k-1)}.$$

Using the relations (5), we obtain the equation

$$bK = aK \cdot uau^{-1} K \cdot u^2 au^{-2} K \cdot \ldots \cdot u^{n/p^k-1} au^{-(n/p^k-1)} K.$$

Further, let us use the following identity:

$$a \cdot u_0 a u_0^{-1} \cdot u_0^2 a u_0^{-2} \cdot \ldots \cdot u_0^{n/p^k-1} a u_0^{-(n/p^k-1)} = (au_0)^{n/p^k} \cdot u_0^{-n/p^k}.$$

Since n divides p^r, then n/p^k is divisible by p^{r-k}. By the equation $u_0^{p^{r-k}} K = K$, we finally obtain $bK = (au_0)^{n/p^k} K$. But we have $bK = K$, then $(au_0)^{n/p^k} K = K$. By the choice of the element a the element au_0K has an order n in the group F/K. Hence we obtain a contradiction. □

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