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M a t h e m a t i c s

TRANSITIVE HYPERIDENTITY IN SEMIGROUPS

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In this paper we characterize all semigroups in which the hyperidentity of transitivity $X(X(x,y),X(y,z)) = X(x,z)$ is polynomially satisfied. In particular, we show that every transitive semigroup (that is a semigroup with the identity $xy^2z = xz$) is also hypertransitive.

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Introduction. A hyperidentity is a second order formula of the form:

$$\forall X_1,\ldots,X_m \forall x_1,\ldots,x_n \quad (\omega_1 = \omega_2), \tag{$*$}$$

where ω_1, ω_2 are words(terms) in the alphabet of functional variables $X_1,\ldots,X_m$ and objective variables $x_1,\ldots,x_n$ (see [1–4]). However, hyperidentities are usually presented without universal quantifiers, i.e. as the equality: $\omega_1 = \omega_2$. The hyperidentity $\omega_1 = \omega_2$ is said to be satisfied in the algebra (Q,Σ), if the equality is true when any functional variable X_i is replaced by any operation of the same arity from Σ (the possibility of such replacements is assumed) and any objective variable x_j is replaced by any element of Q (see also [5,6]).

The variety V satisfies a given hyperidentity, if every algebra of the variety V satisfies the same hyperidentity. Then, the hyperidentity is called hyperidentity of the variety V.

The hyperidentity $(*)$ is said to be non-trivial if $m > 1$, and trivial if $m = 1$. The number m is called the functional rank of the hyperidentity $(*)$.

Let $Q(\cdot)$ be a semigroup. The following function is said to be its binary polynomial (term):

$$F(x,y) = z_1^{\varepsilon_1} z_2^{\varepsilon_2} \ldots z_n^{\varepsilon_n}, \tag{1}$$

where $n \in \mathbb{N}$, $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n \in \mathbb{N}$, $z_1,z_2,\ldots,z_n \in \{x,y\}$ and $z_i \neq z_{i+1}$. The number n is called the length of this representation of the polynomial $F(x,y)$. However,

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due to the identities in the semigroup $Q(\cdot)$, the same polynomial $F(x,y)$ can have different representations of the form (1). If $x^{\delta_1}, x^{\delta_2}, \ldots, x^{\delta_m}$ are all the occurrences of the variable x in the given representation of the polynomial $F(x,y)$, then the sum: $\delta_1+\delta_2+\cdots+\delta_m=\sum\limits_{k=1}^{m}\delta_k$ is called degree of x in $F(x,y)$ and is denoted by $\deg_x(F)$. Analogously, one defines the notion of a degree of the variable y in the representation of $F(x,y)$ and denotes it by $\deg_y(F)$.

Let denote Q^2_{pol} the collection of all binary polynomials of the semigroup $Q(\cdot)$.

We say that the hyperidentity $(*)$ is polynomially satisfied in the semigroup $Q(\cdot)$, if this hyperidentity is satisfied in the binary algebra $\left(Q, Q^2_{pol}\right)$. The semigroup is called hyperassociative, if the following associative trivial hyperidentity

$$X(x, X(y,z)) = X(X(x,y),z) \qquad (**)$$

is polynomially satisfied in this semigroup.

In [7] it is proved that the class of all hyperassociative semigroups forms a finitely based variety, and a basis containing about 1000 identities. In [8] (see also [9]) a basis of the identities of the same variety is given, which contains 5 identities (see also [10–14]).

The semigroup is called hypertransitive, if the following $(***)$ transitive trivial hyperidentity is polynomially satisfied in this semigroup (on the classification of non-trivial transitive hyperidentities see [1–4]):

$$X(X(x,y), X(y,z)) = X(x,z). \qquad (***)$$

Definition 1. A semigroup $Q(\cdot)$ is called transitive, if it satisfies the following identity: $xy^2z = xz$.

In the main result of the present paper we prove that every transitive semigroup is hypertransitive.

Auxiliary Results. Here we prove certain lemmas used in the proof of the main result of the paper.

Lemma 1. Assume $Q(\cdot)$ is a transitive semigroup. Then the following identities are satisfied in Q: $\quad xy^3 = x^3y = xy, \quad x^2yx = xyx^2.$

Proof. The first identity is easily seen by just taking $z=y$ or $x=y$ in the transitive identity. For the second one, first plug in $y=xz$ and get $x(xzxz)z = xz \implies x^2zxz^2 = xz$. Then take $z = yx^2$ in the last identity and get $x^2(yx^2)x(yx^2)(yx^2) = xyx^2$. Since the left hand side is equal to $x^2(yx^3)(yx^2y)x^2 = x^2(yx)(y^2)x^2 = x^2y(xy^2x^2) = x^2yx^3 = x^2yx$, we conclude that $x^2yx = xyx^2$. □

Lemma 2. For any polynomial $X(x,y)$ of the transitive semigroup $Q(\cdot)$ one of the following Cases holds:

1. $X(x,y) = x^n$ for some $n \in \{1,2,3\}$,
2. $X(x,y) = y^m$ for some $m \in \{1,2,3\}$,
3. $X(x,y) = x^ny^m$ for some $m,n \in \{1,2\}$,
4. $X(x,y) = y^mx^n$ for some $m,n \in \{1,2\}$,
5. $X(x,y) = xyx^n$ for some $n \in \{1,2\}$,
6. $X(x,y) = yxy^m$ for some $m \in \{1,2\}$.

P r o o f . Fix a representation for the polynomial $X(x,y)$ having the smallest possible length. Note that the transitive identity implies $x^4 = x^2$. Thus, if the length of the fixed presentation of $X(x,y)$ is 1, we end up with either Case 1 or Case 2.

Since we also have that $x^3y = xy$ and $xy^3 = xy$ from the preceding lemma, any polynomial having a representation with length bigger than 1, has an equivalent presentation of the same length with exponents 1 and 2 only. Therefore, if representation of polynomial $X(x,y)$ has length 2, then it is equivalent to either Case 3 or 4.

Here, we assume that the fixed representation is of the length at least 3 with exponents 1 or 2. Moreover, we can assume that X starts with the variable x and we want to prove that it is $x^{\varepsilon_1}y^{\varepsilon_2}x^{\varepsilon_3}\ldots$ Note that ε_2 cannot be 2, since otherwise we could have a shorter representation. Therefore, we can assume $\varepsilon_2 = 1$. By the preceding lemma, we have $x^2yx = xyx^2$, which allows us to assume $\varepsilon_1 = 1$. Now, if $X(x,y)$ has length 3, then we end up with Case 5.

Since $xyx^2y = xy^2$ and $xyxy = xyx^3y = (xyx^2)xy = (x^2yx)xy = x^2(yx^2y) = x^2y^2$, we conclude that any representation of a polynomial having length 4 or more can be shortened and this completes the proof of the lemma. □

L e m m a 3 . Let $X(x,y)$ be a polynomial with a representation having length bigger than 2 and starting and ending with the variable x in a transitive semigroup. Then $X(x,y) = xy^{\deg_y(X)}x^{\deg_x(X)-1}$.

P r o o f . First, we assume that the powers of the variables are less than 3. If there are more than one instances of y in the representation, then we can eliminate one of them as follows. If one of them has power 2, then we can eliminate y, using the transitive identity. If, otherwise, we have both powers of any two y's are 1, then using the identity $yzy = y^2zy^2$, we replace the powers with 2's and apply the previous case. In these procedures the degree of x is not changed.

Therefore, we can reduce $X(x,y)$ to a representation $x^{\varepsilon_1}y^{\varepsilon_2}x^{\varepsilon_3}$. Since ε_2 has the same parity as the degree of y in $X(x,y)$, we write

$$X(x,y) = x^{\varepsilon_1}y^{\deg_y(X)}x^{\varepsilon_3} = xy^{\deg_y(X)}x^{\deg_x(X)-1}. \qquad \square$$

L e m m a 4 . Let $X(x,y)$ be a polynomial with a representation having length bigger than 1 and starting with the variable x and ending with y in a transitive semigroup. Then $X(x,y) = x^{\deg_x(X)}y^{\deg_y(X)}$.

The proof of this Lemma is similar to the previous one.

Transitive Hyperidentity in Semigroups. We consider the following hyperidentity with functional rank 1:

$$X(X(x,y),X(y,z)) = X(x,z), \tag{2}$$

which is called transitive hyperidentity.

T h e o r e m . The hyperidentity (2) is polynomially satisfied in the semigroup $Q(\cdot)$ if and only if the semigroup $Q(\cdot)$ is transitive, i.e.

$$xy^2z = xz.$$

P r o o f .

Necessity. If we take $X(x,y) = xy$ we get the required transitive identity.

Sufficiency. Due to Lemma 4, it is enough to check the hyperidentity for the polynomials in that lemma.

Claim. It is sufficient to check the hyperidentity for the polynomials starting with x.

Assume we proved the theorem for those polynomials. Let the polynomial $F(x,y)$ start with y. We want to prove $F(F(x,y),F(y,z)) = F(x,z)$. Consider the polynomial $X(x,y) = F(y,x)$ which start with x and, thus, we have
$X(X(z,y),X(y,x)) = X(z,x)$. Since $X(X(z,y),X(y,x)) = F(X(y,x),X(z,y)) =$
$= F(F(x,y),F(y,z))$ and $X(z,x) = F(x,z)$, we get the required identity.

Now, we consider the Cases 1, 3 and 5 of Lemma 4 separately:

Case 1. $X(x,y) = x^n$.

$X(X(x,y),X(y,z)) = X(x^n,y^n) = x^{n^2} = x^n = X(x,z)$, since we have $x^4 = x^2$.

Case 3. $X(x,y) = x^n y^m$.

$X(X(x,y),X(y,z)) = X(x^n y^m, y^n z^m) = (x^n y^m)^n (y^n z^m)^m = (x^{n^2} y^{mn})(y^{mn} z^{m^2})$
$= x^n (y^{mn})^2 z^m = x^n z^m = X(x,z)$, using Lemma 4 and Case 1.

Case 5. $X(x,y) = xyx^n$.

$X(X(x,y),X(y,z)) = X(xyx^n, yzy^n) = (xyx^n)yzy^n(xyx^n)^n = [(xyx^n)y]z[y^n(xyx^n)^n]$
$= x^{n+1}y^2zy^{2n}x^{n(n+1)} = x^{n+1}zx^{n(n+1)} = xzx^{(n+1)^2-1} = xzx^n = X(x,z)$,
using Lemma 4 and Case 1 again. □

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