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Mathematics

ON NOETHERICITY AND INDEX OF DIFFERENTIAL OPERATORS IN ANISOTROPIC WEIGHTED SOBOLEV SPACES

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This paper studies Noethericity and index in anisotropic weighted Sobolev spaces in $\mathbb{R}^m$. Sufficient conditions are established for Noethericity preservation in weighted spaces. Applying the results obtained for operators acting in weighted Sobolev spaces, sufficient condition for semi-elliptic operator to have zero index is found.

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Introduction. In this paper Noethericity preservation problem and the index of differential linear operators in anisotropic weighted Sobolev spaces in $\mathbb{R}^m$ are studied.

Let us state some known results concerning the Noethericity and index of differential operators. Noethericity for elliptic operators on smooth compact manifolds was proved in [1], and the formula for their indices is in the topological form (see [2]). For the elliptic operators in unbounded domains Noethericity has been proved for the special class of operators in weighted Sobolev spaces in $\mathbb{R}^m$ (see [3]), and the Noethericity in terms of limiting operators was studied (see [4]). The class of Noetherian semi-elliptic operators with constant coefficients in $\mathbb{R}^m$ is described in [5, 6], Noethericity for a class of semi-elliptic operators with variable coefficients in weighted Sobolev spaces was obtained in [7]. Index invariance on the scale of anisotropic spaces is studied in [8], where the sufficient condition for it is established.

Basic Concepts and Definitions.

Definition 1. A linear bounded operator A acting from whole a Banach space X to a Banach space Y is called Noetherian, if the following conditions hold:

1. The image of the operator A is closed $\left(Im(A)=\overline{Im(A)}\right)$.
2. The kernel of the operator A is finite dimensional $(\dim Ker(A)<\infty)$.

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3. The cokernel of the operator A is finite dimensional

$$(\dim coker(A) = \dim Y/Im(A) < \infty).$$

The difference between the dimension of the kernel and the cokernel is called index of the operator:

$$\mathrm{ind}(A) = \dim Ker(A) - \dim coker(A).$$

Definition 2. A linear bounded operator A from a Banach space X to a Banach space Y is called normally solvable, if the image of operator A is closed $\left(Im(A) = \overline{Im(A)}\right)$.

Let X_i, Y_i $(i = 1,2)$ be Banach spaces such that X_2 is dense in X_1, Y_2 is dense in Y_1, and the embedding operators $X_2 \subset X_1, Y_2 \subset Y_1$ are bounded.

Let $A_i : X_i \to Y_i$ be bounded linear operators such that $Dom(A_i)|_{X_i} = X_i|_{i=1,2}$ and $A_1 x = A_2 x$, $\forall x \in X_2$. The operators $A_i^* : (Y_i)^* \to (X_i)^*$ are corresponding adjoint operators. Suppose that $A_i : X_i \to Y_i$ $(i = 1,2)$ are Noetherian operators. Notice that $Ker(A_2) \subset Ker(A_1), Ker((A_1)^*) \subset Ker((A_2)^*)$. From Noethericity of A_i it follows that $\dim coker(A_i) = \dim Ker((A_i)^*)$ (see [9]). So the following inequalities hold:

$$\dim Ker(A_1) \geq \dim Ker(A_2), \ \dim coker(A_2) \geq \dim coker(A_1). \tag{1}$$

From (1) it follows that $\mathrm{ind}(A_1) \geq \mathrm{ind}(A_2)$. So $\mathrm{ind}(A_1) = \mathrm{ind}(A_2)$ holds if and only if

$$\dim Ker(A_1) = \dim Ker(A_2), \ \dim coker(A_2) = \dim coker(A_1). \tag{2}$$

Let $m \in \mathbb{N}$, $\mathbb{Z}_+^m$, $\mathbb{N}^m$, $\mathbb{R}^m$ be sets of m-dimensional: multi-indices, multi-indices with natural components and Euclidean space respectively.

Set

$$Q := \left\{ g(x) \in C^\infty(\mathbb{R}^m) : g(x) > 0, \forall x \in \mathbb{R}^m; \ \frac{|D^\beta g(x)|}{g(x)} \rightrightarrows 0 \Big|_{|x| \to \infty}, \right.$$
$$\left. \forall \beta \in \mathbb{Z}_+^m, \ \beta \neq 0 \right\}.$$

For $k \in \mathbb{Z}_+$ and $\nu \in \mathbb{N}^m$ denote

$$C^{k,\nu}(\mathbb{R}^m) := \left\{ a(x) : D^\beta a(x) \in C(\mathbb{R}^m), \ \sup_{x \in \mathbb{R}^m} |D^\beta a(x)| < \infty, \right.$$
$$\left. \forall \beta \in \mathbb{Z}_+^m, \ s.t. \ (\beta : \nu) \leq k \right\}.$$

Definition 3. For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^m$ denote by $H^{k,\nu}(\mathbb{R}^m)$ the space of measurable functions $\{u\}$ equipped with a norm

$$\|u\|_{k,\nu} = \left(\sum_{(\alpha:\nu) \leq k} \int |D^\alpha u(x)|^2 dx \right)^{1/2} < \infty.$$

Definition 4. For $k \in \mathbb{Z}_+,\ \nu \in \mathbb{N}^m$ and for a positive function $q(x)$ denote by $H_q^{k,\nu}(\mathbb{R}^m)$ the space of measurable functions $\{u\}$ equipped with a norm

$$\|u\|_{k,\nu,q} = \left(\sum_{(\alpha:\nu)\le k} \int \left| D^\alpha u(x)\, q(x)^{(k-(\alpha:\nu))} \right|^2 dx \right)^{1/2} < \infty.$$

Definition 5. For $k \in \mathbb{Z}_+, \nu \in \mathbb{N}^m$ and $\mu \in Q$ denote by $\widetilde{H}_\mu^{k,\nu}(\mathbb{R}^m)$ the space of functions u with $\mu u \in H^{k,\nu}(\mathbb{R}^m)$ with a norm

$$\|u\|'_{k,\nu,\mu} = \|\mu u\|_{k,\nu} < \infty.$$

Consider $k, s \in \mathbb{N}$ and $k \ge s$.

Let

$$P(x,\mathbb{D}) = \sum_{(\alpha:\nu)\le s} a_\alpha(x) D^\alpha, \tag{3}$$

where $m \in \mathbb{N}; \alpha \in \mathbb{Z}_+^m;\ \ \nu \in \mathbb{N}^m;\ \ (\alpha:\nu) = \frac{\alpha_1}{\nu_1} + \ldots + \frac{\alpha_m}{\nu_m};\ \ s \in \mathbb{N}$, $D^\alpha = D_1^{\alpha_1} \ldots D_m^{\alpha_m};\ D_j = -i\frac{\partial}{\partial x_j};\ x = (x_1, \ldots, x_m) \in \mathbb{R}^m;\ \ a_\alpha(x) \in C^{k-s,\nu}(\mathbb{R}^m)$.

Denote the principal part of $P(x,\mathbb{D})$ and its symbol by

$$P_s(x,\mathbb{D}) = \sum_{(\alpha:\nu)=s} a_\alpha(x) D^\alpha, \quad P_s(x,\xi) = \sum_{(\alpha:\nu)=s} a_\alpha(x)\xi^\alpha. \tag{4}$$

With certain conditions on the coefficients of differential form $P(x,\mathbb{D})$ it defines a linear bounded operator acting from whole $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$. Denote this operatorby $\left(P; H^{k,\nu}\right)$.

The differential form $P(x,\mathbb{D})$ defines a bounded linear operator acting from whole $\widetilde{H}_\mu^{k,\nu}(\mathbb{R}^m)$ to $\widetilde{H}_\mu^{k-s,\nu}(\mathbb{R}^m)$. Denote it by $\left(P; \widetilde{H}_\mu^{k,\nu}\right)$.

For a function $q(x)$ with $\frac{1}{q(x)} \rightrightarrows 0\Big|_{|x|\to\infty}$, the differential form $P(x,\mathbb{D})$ defines a bounded linear operator acting from whole $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$. Denote this operator by $\left(P; H_q^{k,\nu}\right)$.

Definition 6. The differential expression $P(x,D)$ of the form (3) is called semi-elliptic at a point $x = x_0$, if the following is satisfied:

$$P_s(x_0,\xi) \ne 0;\ \forall \xi \in \mathbb{R}^m;\ |\xi| \ne 0.$$

Definition 7. The differential expression $P(x,D)$ of the form (3) is called semi-elliptic in $\mathbb{R}^m$ or just semi-elliptic, if it is semi-elliptic at each point $x \in \mathbb{R}^m$.

Main Results.

Lemma 1. The operator $\left(P; H^{k,\nu}\right)$ is a Noetherian operator if and only if $\left(P; \widetilde{H}_\mu^{k,\nu}\right)$ is Noetherian, and the following equalities hold:

$$\dim Ker\left(P; H^{k,\nu}\right) = \dim Ker\left(P; \widetilde{H}_\mu^{k,\nu}\right),$$

$$\dim coker\left(P; H^{k,\nu}\right) = \dim coker\left(P; \widetilde{H}_\mu^{k,\nu}\right),$$

$$\operatorname{ind}\left(P; H^{k,\nu}\right) = \operatorname{ind}\left(P; \widetilde{H}_\mu^{k,\nu}\right).$$

P r o o f . Let M_μ be the operator of multiplication by $\mu(x)$:
$M_\mu : \widetilde{H}^{k,\nu}_\mu(\mathbb{R}^m) \to H^{k,\nu}(\mathbb{R}^m),\ M_\mu u(x) = \mu(x)u(x),\ \forall u \in \widetilde{H}^{k,\nu}_\mu(\mathbb{R}^m)$;
$M^{-1}_\mu : H^{k,\nu}(\mathbb{R}^m) \to \widetilde{H}^{k,\nu}_\mu(\mathbb{R}^m),\ M^{-1}_\mu v(x) = v(x)/\mu(x),\ \forall v \in H^{k,\nu}(\mathbb{R}^m)$.
Consider the following operator: $\tilde{P}u \equiv M_\mu P M^{-1}_\mu$.

It is a linear bounded operator acting from $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$. Then for $u \in H^{k,\nu}(\mathbb{R}^m)$ we have $\tilde{P}u = M_\mu P M^{-1}_\mu u = Pu + Tu$, where

$$Tu = \sum_{(\alpha:\nu)\le s} a_\alpha(x) \sum_{\beta\le\alpha,\beta\neq 0} C^\beta_\alpha \mu(x) D^\beta\left(\frac{1}{\mu(x)}\right) D^{\alpha-\beta}u(x).$$

The operator $T : H^{k,\nu}(\mathbb{R}^m) \to H^{k-s,\nu}(\mathbb{R}^m)$ is linear bounded with lower order terms, and for each $0 \neq \beta \in \mathbb{Z}^m_+$ we have $\mu(x)D^\beta\left(\frac{1}{\mu(x)}\right) \rightrightarrows 0\Big|_{|x|\to\infty}$.

Taking into account the above remarks and conditions on the coefficients of the operator, it can be checked that for each $\varepsilon > 0$ there exists $\phi_\varepsilon(x) \in C^\infty_0(\mathbb{R}^m)$ such that $T = T^{'}_\varepsilon + T^{''}_\varepsilon$, where $T^{'}_\varepsilon = (1-\phi_\varepsilon)T$ satisfies

$$\|T^{'}_\varepsilon u\|_{k-s,\nu} \le \varepsilon\|u\|_{k,\nu},\ \forall u \in H^{k,\nu}(\mathbb{R}^m),$$

and $T^{''}_\varepsilon = \phi_\varepsilon T$ is a compact operator acting from $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$.

So, applying the Theorem 8.3.2 from [10], we get that $T : H^{k,\nu}(\mathbb{R}^m) \to H^{k-s,\nu}(\mathbb{R}^m)$ is a compact operator.

This implies that $\tilde{P}(x,\mathbb{D}) : H^{k,\nu}(\mathbb{R}^m) \to H^{k-s,\nu}(\mathbb{R}^m)$ is Noetherian operator if and only if $P(x,\mathbb{D}) : H^{k,\nu}(\mathbb{R}^m) \to H^{k-s,\nu}(\mathbb{R}^m)$ is Noetherian with index equality

$$\mathrm{ind}\left(\tilde{P};H^{k,\nu}\right) = \mathrm{ind}\left(P;H^{k,\nu}\right) \quad \text{(see [10], 8.5.20).}$$

Consider $u \in Ker\left(\tilde{P};H^{k,\nu}\right)$. Then it is easy to see that $v = M^{-1}_\mu u \in \widetilde{H}^{k,\nu}_\mu(\mathbb{R}^m)$ and $Pv = 0$. On the other hand, for $v \in Ker\left(P;\widetilde{H}^{k,\nu}_\mu\right)$ we have $u = M_\mu v \in H^{k,\nu}(\mathbb{R}^m)$ and $\tilde{P}u = 0$. Considering similar correspondence for adjoint operator's kernel elements, we get that there are bijections between bases of kernels and adjoint operator's kernels for $\left(\tilde{P};H^{k,\nu}\right)$ and $\left(P;\widetilde{H}^{k,\nu}_\mu\right)$, so,

$$\dim Ker\left(P;\widetilde{H}^{k,\nu}_\mu\right) = \dim Ker\left(\tilde{P};H^{k,\nu}\right), \tag{5}$$

$$\dim Ker\left(P;\widetilde{H}^{k,\nu}_\mu\right)^* = \dim Ker\left(\tilde{P};H^{k,\nu}\right)^*. \tag{6}$$

If $\tilde{P}(x,\mathbb{D})$ is Noetherian, then from its normal solvability we get $Im\left(\tilde{P};H^{k,\nu}\right) = {}^\perp\left(Ker\left(\tilde{P};H^{k,\nu}\right)^*\right)$ (see [9]), where ${}^\perp\left(Ker\left(\tilde{P};H^{k,\nu}\right)^*\right)$ is the set of elements from $H^{k-s,\nu}(\mathbb{R}^m)$, which are orthogonal to $Ker\left(\tilde{P};H^{k,\nu}\right)^*$. Using this, it can be shown that for $\left(P;\widetilde{H}^{k,\nu}_\mu\right)$ holds $Im\left(P;\widetilde{H}^{k,\nu}_\mu\right) = {}^\perp\left(Ker\left(P;\widetilde{H}^{k,\nu}_\mu\right)^*\right)$. It implies that $\left(P;\widetilde{H}^{k,\nu}_\mu\right)$ is also normally solvable $\left(Im\left(P;\widetilde{H}^{k,\nu}_\mu\right) = \overline{Im\left(P;\widetilde{H}^{k,\nu}_\mu\right)}\right)$. Similarly it can be shown, that if $\left(P;\widetilde{H}^{k,\nu}_\mu\right)$ is Noetherian, then from its normal solvability follows the normal solvability of $\left(\tilde{P};H^{k,\nu}\right)$. And taking into account (5),(6), we obtain that from Noethericity of $\left(\tilde{P};H^{k,\nu}\right)$ it follows that $\left(P;\widetilde{H}_{\mu}^{k,\nu}\right)$ is Noetherian and vice versa. From (5),(6) for indexes we get $\operatorname{ind}\left(\tilde{P};H^{k,\nu}\right)=\operatorname{ind}\left(P;\widetilde{H}_{\mu}^{k,\nu}\right)$. So it is proved that $\left(P;H^{k,\nu}\right)$ is a Noetherian operator if and only if $\left(P;\widetilde{H}_{\mu}^{k,\nu}\right)$ is Noetherian, and we have $\operatorname{ind}\left(P;H^{k,\nu}\right)=\operatorname{ind}\left(P;\widetilde{H}_{\mu}^{k,\nu}\right)$. From (2) it follows that

$$\dim Ker\left(P;H^{k,\nu}\right)=\dim Ker\left(P;\widetilde{H}_{\mu}^{k,\nu}\right),$$
$$\dim coker\left(P;H^{k,\nu}\right)=\dim coker\left(P;\widetilde{H}_{\mu}^{k,\nu}\right).$$

□

L e m m a 2. Let $q(x)\in Q$ be a function satisfying $\dfrac{1}{q(x)}\rightrightarrows 0\Big|_{|x|\to\infty}$. Let $\left(P;H^{k,\nu}\right)$ be a Noetherian operator and $\left(P;H_{q}^{k,\nu}\right)$ be normally solvable. Then $\left(P;H_{q}^{k,\nu}\right)$ is also Noetherian with

$$\dim Ker\left(P;H_{q}^{k,\nu}\right)=\dim Ker\left(P;H^{k,\nu}\right),$$
$$\dim coker\left(P;H_{q}^{k,\nu}\right)=\dim coker\left(P;H^{k,\nu}\right),$$
$$\operatorname{ind}\left(P;H_{q}^{k,\nu}\right)=\operatorname{ind}\left(P;H^{k,\nu}\right).$$

P r o o f . Consider $\mu(x)=(q(x))^{k}\in Q$. Then we will have the following embeddings:

$$\widetilde{H}_{\mu}^{k,\nu}(\mathbb{R}^{m})\hookrightarrow H_{q}^{k,\nu}(\mathbb{R}^{m})\hookrightarrow H^{k,\nu}(\mathbb{R}^{m}).$$

Considering also the embeddings for adjoint spaces, from Lemma 1 we get

$$\dim Ker\left(P;\widetilde{H}_{\mu}^{k,\nu}\right)=\dim Ker\left(P;H_{q}^{k,\nu}\right)=\dim Ker\left(P;H^{k,\nu}\right)<\infty, \quad (7)$$

$$\dim Ker\left(P;\widetilde{H}_{\mu}^{k,\nu}\right)^{*}=\dim Ker\left(P;H_{q}^{k,\nu}\right)^{*}=\dim Ker\left(P;H^{k,\nu}\right)^{*}<\infty. \quad (8)$$

Since $\left(P;H_{q}^{k,\nu}\right)$ is normally solvable and $\dim Ker\left(P;H_{q}^{k,\nu}\right)^{*}<\infty$, we obtain $\dim coker\left(P;H_{q}^{k,\nu}\right)=\dim Ker\left(P;H_{q}^{k,\nu}\right)^{*}<\infty$ (see [9]).

From this and (7), (8) it follows that $\left(P;H_{q}^{k,\nu}\right)$ is also Noetherian and $\operatorname{ind}\left(P;H^{k,\nu}\right)=\operatorname{ind}\left(P;H_{q}^{k,\nu}\right)$. □

Let

$$L_{s}(\mathbb{D})=\sum_{(\alpha:\nu)=s}a_{\alpha}D^{\alpha}, \quad (9)$$

where the coefficients a_{α} are real numbers and suppose the the same relations after Eq. (3) are satisfied.

Denote by

$$T(x,\mathbb{D})=\sum_{(\alpha:\nu)<s}b_{\alpha}(x)D^{\alpha} \quad (10)$$

the lower order terms of differential form, where the same notations are used as for Eq. (3) and $b_{\alpha}(x)\in C^{k-s,\nu}(\mathbb{R}^{m})$.

For a function $q(x)$ satisfying $\frac{1}{q(x)} \rightrightarrows 0\big|_{|x|\to\infty}$ it can be checked that $T(x,\mathbb{D})$ gives a linear bounded operator, acting from $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$. Applying Theorem 8.3.2. from [10] for $(T;H_q^{k,\nu})$, following lemma can be obtained.

L e m m a 3. Let function $q(x)$ satisfy $\frac{1}{q(x)} \rightrightarrows 0\big|_{|x|\to\infty}$. Then the operator $(T;H_q^{k,\nu})$ is a compact operator.

Consider the operator $L(x,\mathbb{D}) = L_s(\mathbb{D}) + T(x,\mathbb{D})$. It generates a linear bounded operator acting from $H^{k,\nu}(\mathbb{R}^m)$ to $H^{k-s,\nu}(\mathbb{R}^m)$ $\left(\text{it is denoted by } (L;H^{k,\nu})\right)$ and for function $q(x)$, which satisfies $1/q(x) \rightrightarrows 0\big|_{|x|\to\infty}$, originates linear bounded operator acting from $H_q^{k,\nu}(\mathbb{R}^m)$ to $H_q^{k-s,\nu}(\mathbb{R}^m)$ $\left(\text{denoted by} (L;H_q^{k,\nu})\right)$.

T h e o r e m . Let $q(x) \in Q$ be a function, with $\frac{1}{q(x)} \rightrightarrows 0\Big|_{|x|\to\infty}$. Let $\left(L;H^{k,\nu}\right)$ be a semi-elliptic Noetherian operator and $\left(L;H_q^{k,\nu}\right)$ be normally solvable. Then $\operatorname{ind}\left(L;H^{k,\nu}\right) = 0$.

P r o o f . Appling Lemma 2, we conclude that $\left(L;H_q^{k,\nu}\right)$ is also Noetherian and $\operatorname{ind}\left(L;H^{k,\nu}\right) = \operatorname{ind}\left(L;H_q^{k,\nu}\right)$. Then due to semi-ellipticity of $L(x,\mathbb{D})$ and the fact that coefficients a_α of its principal part are reals, there exists c_0 such that $L(x,\mathbb{D})$ can be represented as

$$L(x,\mathbb{D}) = L^1(\mathbb{D}) + L^2(x,\mathbb{D}),$$

where $L^1(\mathbb{D}) = L_s(\mathbb{D}) + c_0$, $L^1(\xi) \neq 0$, $\forall \xi \in \mathbb{R}^m$ and $L^2(x,\mathbb{D}) = T(x,\mathbb{D}) - c_0$.

From Lemma 3 we get that $\left(L^2;H_q^{k,\nu}\right)$ is a compact operator. It follows that $\left(L^1;H_q^{k,\nu}\right)$ is Noetherian and $\operatorname{ind}\left(L;H_q^{k,\nu}\right) = \operatorname{ind}\left(L^1;H_q^{k,\nu}\right)$ (see [10], 8.5.20). In [5] it is proved that $L^1(\mathbb{D}) : H^{k,\nu}(\mathbb{R}^m) \to H^{k-s,\nu}(\mathbb{R}^m)$ is a Noetherian operator and $\operatorname{ind}\left(L^1;H^{k,\nu}\right) = 0$. Lemma 2 can be applied for $L^1(\mathbb{D})$ and we get

$$\operatorname{ind}\left(L^1;H_q^{k,\nu}\right) = \operatorname{ind}\left(L^1;H^{k,\nu}\right) = 0.$$

So, $\operatorname{ind}\left(L;H^{k,\nu}\right) = \operatorname{ind}\left(L;H_q^{k,\nu}\right) = \operatorname{ind}\left(L^1;H_q^{k,\nu}\right) = \operatorname{ind}\left(L^1;H^{k,\nu}\right) = 0.$ □

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