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Mathematics

ON DISTRIBUTION'S CONSTANT SLOWLY VARYING COMPONENT

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In the present report it is proved that for a priori given numbers $\rho \in (1, +\infty)$ and $L \in R^+ = (0, +\infty)$ there is a distribution $\{p_n\}_1^\infty$ with the following properties: $\{p_n\}_1^\infty$ varies regularly as $n \to +\infty$ with exponent $(-\rho)$, exhibits the constant slowly varying component L, and $\{\log p_n\}_1^\infty$ is downward convex.

Keywords**:** distribution, regular variation, constant slowly varying component.

1⁰. Let $\{p_n\}_1^\infty$ be a regularly varying as $n \to +\infty$ distribution with exponent $(-\rho)$, $1 \le \rho < +\infty$, i.e. for $s = 2,3,...$ the limit exists $\lim\limits_{n\to+\infty} (p_{s\cdot n} / p_n) = s^{-\rho}$ (see [1]).

There is a slowly varying sequence $\{L(n)\}_1^\infty$, i.e. $L(n) > 0$, $n = 1,2,...$, and for $s = 2,3,...$ the limit exists $\lim\limits_{n\to+\infty} (L(s \cdot n) / L(n)) = 1$ such that

$$p_n \approx n^{-\rho} L(n),\ n \to +\infty \tag{1}$$

(we write $f_n \approx g_n$, $n \to +\infty$ for $\{f_n\}$ and $\{g_n\}$, if $\lim\limits_{n\to+\infty} (f_n / g_n) = 1$).

If for $L(n)$ in (1) the limit $\lim\limits_{n\to+\infty} L(n) = L \in R^+ = (0, +\infty)$ exists, then we say that $\{p_n\}_1^\infty$ exhibits a constant slowly varying component (CSVC).

In Bioinformatics there is a restriction on distribution of type (1): *the graph of* $\{\log p_n\}_1^\infty$ *consists of at most three upward/downward convex pieces* (see [2]).

In the present report we establish the following

Theorem. Given the constants $\rho \in (1, +\infty)$ and $L \in R^+$. There is a distribution $\{p_n\}_1^\infty$, which :

a) varies regularly as $n \to +\infty$ with exponent $(-\rho)$;

b) exhibits CSVC L;

c) generates the sequence $\{\log p_n\}_1^\infty$ with graph consisting of one downward convex piece.

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The proof is based on special distribution of the type

$$p_n = c(\rho)n^{-\rho}\,,\ n = 1,2,\ldots\,,\ c(\rho) = (\sum_{n\ge 1} n^{-\rho})^{-1}\,, \tag{2}$$

where $(-\rho)$ presents the exponent of regular variation and $c(\rho)$ is it's CSVC.

For distribution of type (2) the sequence $\{\log p_n\}_1^\infty$ satisfies statement c) of the Theorem. Indeed, we have to verify the inequality $\log p_n - \log p_{n+1} < \log p_{n+1} - \log p_{n+2}$ for index $n \ge 1$. The latter inequality is equivalent to $(p_n / p_{n+1}) > (p_{n+1} / p_{n+2})$ or due to (2), to $((n+1)/n)^\rho > ((n+2)/(n+1))^\rho$ that leads to $1+(1/n) > 1+(1/(n+1))$.

Hence if the equality $L = c(\rho)$ holds for given ρ and L, the distribution $\{p_n\}_1^\infty$ satisfying the Theorem is constructed.

Thus, there remains the case $L \ne c(\rho)$, where $c(\rho)$ is given by formula (2).

2⁰. Consider the continuous analogue of the sequence $q_n = Ln^{-\rho}$, $n = 1,2,\ldots$:

$$f(t) = Lt^{-\rho}\,,\ t \in [1,+\infty)\,. \tag{3}$$

Let us draw the tangent line to the curve $y = f(t)$ at the entire point $n_0 > 1$. Choice of this point will be done later. Since $f'(t) = -\rho L t^{-\rho-1}$, $t \in [1,+\infty)$, and for the tangent line $y(t) = at + b$ to the curve $y = f(t)$ at point $t = n_0$ we have $y(n_0) = f(n_0)$, $a = f'(n_0)$, therefore $b = Ln_0^{-\rho}(1+\rho)$ and

$$y(t) = Ln_0^{-\rho}\left\{-\frac{\rho t}{n_0} + (1+\rho)\right\}. \tag{4}$$

The finite sum

$$\sum_{k=1}^{n_0} y(k) = Ln_0^{-\rho}\sum_{k=1}^{n_0}\left\{-\frac{\rho k}{n_0} + (1+\rho)\right\} = L\frac{1}{n_0^{\rho-1}}\left\{\frac{\rho}{2} + 1 - \frac{\rho}{2n_0}\right\} \tag{5}$$

is evaluated easily with the help of (4). Then

$$\sum_{k=1}^{n_0} y(k) + \sum_{n>n_0} q_n = L\frac{1}{n_0^{\rho-1}}\left\{\frac{\rho}{2} + 1 - \frac{\rho}{2n_0}\right\} + L\sum_{n>n_0} n^{-\rho} \overset{def}{=} T_{n_0}\,. \tag{6}$$

Since $\rho \in (1,+\infty)$, then for n_0 large enough we may get the inequality

$$T_{n_0} < 1\,. \tag{7}$$

Let $\{e_n\}_1^{n_0}$ be a decreasing sequence of non-negative numbers with $e_k > 0$, $k = 1,2,\ldots,n_0-1$, $e_{n_0} = 0$, for which $\{\log e_n\}_1^{n_0}$ is downward convex and

$$\sum_{k=1}^{n_0} e_k = 1 - T_{n_0}\,. \tag{8}$$

Here T_{n_0} is given by equality (6). Let us give an example of such a sequence.

Example. Put $e_n = M\left(\frac{1}{n} + \frac{1}{n_0}\right)$, $n = 1,2,\ldots,n_0$, where M is a positive constant. $\{e_n\}_1^{n_0}$ decreases and $e_{n_0} = 0$. The downward convexity of $\{\log e_n\}_1^{n_0}$ is proved similarly to the case (2). The constant M is defined uniquely from the condition

$$M\sum_{k=1}^{n_0-1}\left(\frac{1}{n}+\frac{1}{n_0}\right)=1-T_{n_0}\,.$$

For $L\neq c(\rho)$ the distribution $\{p_n\}_1^{\infty}$, satisfying Theorem, is built as follows:

$$p_k=\begin{cases} y(k)+e_k & \text{for} \quad k=1,2,\dots,n_0,\\ q_k & \text{for} \quad k>n_0.\end{cases} \tag{9}$$

It is clear that $\{p_n\}_1^{\infty}$, defined by equalities (9), is a distribution, because by (6)–(8) we have $\sum\limits_{k\geq 1}p_k=\sum\limits_{k=1}^{n_0}(y(k)+e_k)+\sum\limits_{n>n_0}q_n=1$.

The distribution $\{p_k\}_1^{\infty}$ of type (9) varies regularly as $n\to+\infty$ with exponent $(-\rho)$ and exhibits CSVC because $p_n\approx q_n=Ln^{-\rho}$, $n\to+\infty$. Here we used (9).

Finally, the sequence $\{y(k)+e_k\}_{k=1}^{n_0}$ being generated by the sequence $\{\log(y(k)+e_k)\}_{k=1}^{n_0}$ becomes downward convex for n_0 large enough. Note that n_0 is the point, to which the tangent line was drawn.

Indeed, according to (4), for n_0 large enough the number c, where $0<c\overset{def}{=}y(k)-y(k+1)$, $k=1,2,\dots,n_0-1$ ($y(t)$ is linear), may be made arbitrary small.

That is why we may choose n_0 in order to get inequalities

$$2ce_k+e_{k+2}>2ce_{k+1}\,,\ k=1,2,\dots,n_0-1\,. \tag{10}$$

Let us take n_0 so large that the inequalities (7) and (10) take place and fix n_0. Let us prove the validity of inequalities

$$y(k)e_{k+2}+y(k+2)e_k>2y(k+1)e_{k+1}\,,\ k=1,2,\dots,n_0-1\,, \tag{11}$$

using (10). Since $y(k+1)=y(k)+c$, $y(k+2)=y(k)+2c$, then (11) may be written in the form

$$y(k)(\mathring{a}_{\hat{e}+2}+\mathring{a}_k-2e_{k+1})+2c(e_k-e_{k+1})>0\,,\ k=1,2,\dots,n_0-1\,. \tag{12}$$

Since the sequence $\{\log e_k\}_1^{n_0}$ is downward convex, then due to [3] $\{e_k\}_1^{n_0}$ is downward convex. That is why the first term at the left-hand-side of (12) is positive. Now (12) follows from the decrease of sequence $\{e_k\}_1^{n_0}$. Thus (11) is proved.

The log-downward convexity of sequences $\{y(k)\}_1^{n_0}$ and $\{e_k\}_1^{n_0}$ means that the there hold the following inequalities

$$y(k)y(k+2)>(y(k+1))^2\,,\ e_ke_{k+2}>e_{k+1}^2\,,\ k=1,2,\dots,n_0-1\,.$$

Summing up these inequalities with (11), we obtain for $k=1,2,\dots,n_0-1$

$$y(k)y(k+2)+y(k)e_{k+2}+y(k+2)e_k+e_ke_{k+2}>(y(k+1))^2+2y(k+1)e_{k+1}+e_{k+1}^2\,.$$

Last inequalities are easily transformed into

$$\frac{y(k)+e_k}{y(k+1)+e_{k+1}} > \frac{y(k+1)+e_{k+1}}{y(k+2)+e_{k+2}}, \ k=1,2,...,n_0-1,$$

which prove the statement for these indices.

Returning to (9), we become certain that $\{\log p_k\}_1^\infty$ is downward convex, because for indices $n_0, n_0+1,...$ the statement is obvious.

Theorem is proved.

Remark. It is easy to see that the constructed distribution $\{p_n\}$ of type (9) is downward convex (see [3]).

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Գ. Պ. Ավագյան

Հաստատուն դանդաղ փոփոխվող բաղադրիչով բաշխման մասին

Աշխատանքում ապացուցված է, որ նախապես տրված $\rho \in (1,+\infty)$ և $L \in R^{+} = (0,+\infty)$ թվերի համար գոյություն ունի հետևյալ հատկություններով օժտված $\{p_n\}_1^{\infty}$ բաշխում: Այն կանոնավոր է փոփոխվում $(-\rho)$ ցուցիչով, երբ $n \to +\infty$, ունի L հաստատուն դանդաղ փոփոխվող բաղադրիչ, և $\{\log p_n\}_1^{\infty}$-ը ուռուցիկ է դեպի ներքև:

Г. П. Авагян.

О постоянной медленно меняющейся компоненте распределения

В сообщении доказано, что для априори заданных чисел $\rho \in (1,+\infty)$ и $L \in R^{+} = (0,+\infty)$ существует распределение $\{p_n\}_1^{\infty}$ со следующими свойствами: $\{p_n\}_1^{\infty}$ правильно меняется при $n \to +\infty$ с показателем $(-\rho)$, допускает постоянную правильно меняющуюся компоненту L и последовательность $\{\log p_n\}_1^{\infty}$ выпукла вниз.