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COMMUNICATIONS

Mathematics

ON UNIQUENESS OF HOLOMORPHIC AND BOUNDED OUTSIDE THE CLOSED LOGARITHMIC SECTOR FUNCTIONS REPRESENTABLE BY LACUNARY POWER SERIES

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In the present note it is shown that for a set of positive integers Λ a Müntz-type condition holds if and only if there exists a lacunary power series $f(z)=\sum_{\nu\in\Lambda} f_\nu / z^\nu$ that allows an analytic and bounded continuation to the complement of a closed logarithmical sector with vertex at the origin.

***Keywords*:** closed logarithmic sector, lacunary power series, coefficient function method, analytic continuation.

Let C_∞ be the extended complex plane. Denote by Δ_β^α the closed logarithmical sector $\Delta_\beta^\alpha=\{z:|z|\le 1,\ |\arg z-\alpha\ln|z||\le\beta\}$ with $\alpha\in R, \beta\in[0,\pi)$. The main result of this note is the following

Theorem. Let Λ be a set of non-negative integers and let $\beta\in[0,\pi), \alpha\in R$. Then there exists a non-trivial holomorphic and bounded function f in $C_\infty\setminus\Delta_\beta^\alpha$ with power series

$$f(z)=\sum_{\nu\in\Lambda}\frac{f_\nu}{z^\nu},\qquad |z|>1, \tag{1}$$

if and only if

$$\limsup_{r\to\infty}\left[\sum_{\substack{\nu\in N\setminus\Lambda\\ \nu\le r}}\frac{1}{\nu}-\frac{\beta}{\pi}\ln r\right]<+\infty. \tag{2}$$

Note that the particular case $\alpha=0$ of the Theorem coincides with Theorem 1 from [1].

The proof of Theorem is based on the representation of holomorphic and bounded on $C_\infty\setminus\Delta_\beta^\alpha$ functions in the form $f(z)=\sum_{\nu=0}^{\infty}\varphi(\nu)/z^\nu$, where φ is a function of exponential type in a certain half plane. This well-known technique of

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«coefficient function» was successfully used in theory of analytic continuation (see [2–4]).

Let $\Pi_\alpha = \{z : \mathrm{Re}\, z \ge \alpha\, \mathrm{Im}\, z,\ \alpha \in R\}$ be the closed half plane. In polar coordinates r, θ the half plane Π_α is represented as

$$\Pi_\alpha = \left\{ z = r e^{i\theta} : r \ge 0, \theta \in \left[\gamma - \frac{\pi}{2}, \gamma + \frac{\pi}{2} \right] \right\},$$

where γ is the root of the equation

$$\mathrm{tg}\gamma = -\alpha,\ \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

Lemma 1. If f is holomorphic and bounded in $C_\infty \setminus \Delta_\beta^\alpha$, then there exists a function φ continuous on Π_α and holomorphic on its interior such that

$$\varphi(w) e^{-\beta|\mathrm{Im}\, w|} = O(1), \qquad w \to \infty, \tag{3}$$

and

$$f(z) = f_0 + \sum_{\nu=0}^{\infty} \frac{\varphi(\nu)}{z^{\nu+1}}, \qquad |z| > 1$$

with some constant f_0.

Lemma 2. Let φ be holomorphic on Π_α and such that for some $\gamma > 1$

$$\varphi(w) e^{-\beta|\mathrm{Im}\, w|} = O\left(\frac{1}{|w|^\gamma} \right), \qquad w \to \infty.$$

Then $\sum_{\nu=0}^{\infty} \frac{\varphi(\nu)}{z^\nu}$ defines a bounded analytic function in $C_\infty \setminus \Delta_\beta^\alpha$.

The proof of Lemmas 1, 2 is similar to the proofs of corresponding lemmas from [1] with some differences.

Proof of the Theorem.

1. Suppose that there is a function f satisfying conditions of the Theorem. According to Lemma 1 we have

$$f(z) = f_0 + \sum_{\mu=0}^{\infty} \frac{\varphi(\mu - 1)}{z^\mu}, \quad |z| > 1,$$

whereas the function $\varphi \ne 0$ is continuous on Π_α, holomorphic on its interior and satisfies $\varphi(w) e^{-\beta|\mathrm{Im}\, w|} = O(1),\ w \to \infty$.

From (1) we get $\varphi(\nu),\ \nu \in N_1$, where $N_1 := N \setminus \Lambda - 1$. Applying Carleman's formula [5] to the function $\varphi(z e^{i\gamma} + a), \mathrm{Re}\, z \ge 0$, where $a \in (0,1)$ is such that $\varphi(a) \ne 0$, we find

$$\cos\gamma \sum_{\substack{\nu \in N_1 \\ \nu \le r+a}} (\nu - a)^{-1} \le \frac{1}{2\pi} \int_1^r (t^{-2} - r^{-2}) \ln \left| \varphi(-it e^{i\gamma} + a) \varphi(it e^{i\gamma} + a) \right| dt +$$

$$+ \frac{1}{\pi r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left| \varphi(r e^{i(\theta+\gamma)} + a) \right| \cos\theta\, d\theta + O(1), \qquad r \to \infty.$$

From (3) it easily follows that the first integral is bounded from above by $\frac{\beta}{\pi}\cos\gamma\ln r + O(1)$ as $r \to \infty$. The second integral is $O(1)$ as $r \to \infty$, therefore we obtain the estimate

$$\limsup_{r\to\infty}\left[\sum_{\substack{\nu\in N_1\\ \nu\le r+a}} (\nu - a)^{-1} - \frac{\beta}{\pi}\ln r\right] < +\infty ,$$

which is equivalent to condition (2).

2. For $\alpha \in R$ choose $\gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\mathrm{tg}\gamma = -\alpha$. Following W.H.J. Fuchs [6], we consider the function

$$\varphi(w) = L^w H_\gamma(we^{-i\gamma})\exp\left(-2\left(1-\frac{\beta}{\pi}\right)\cos\gamma\, we^{-i\gamma}\ln(1+we^{-i\gamma})\right),\ w \in \Pi_\alpha .$$

Here ln is the principal branch of logarithm, $L > 0$ is a constant and

$$H_\gamma(z) = \prod_{\nu\in N\setminus\Lambda} \frac{\nu - z e^{i\gamma}}{\nu + z e^{i\gamma}}\exp\left(\frac{2z}{\nu}\cos\gamma\right).$$

It is well-known that under condition (2) and for L small enough, it holds

$$\varphi(w)e^{-\beta|\mathrm{Im}\, w|} = O\left(\frac{1}{|w+1|^2}\right), \qquad w \to \infty .$$

If f is given by

$$f(z) = \sum_{\nu=0}^{\infty}\frac{\varphi(\nu)}{z^\nu}, \quad |z| > 1 ,$$

then, due to Lemma 2, f defines a bounded analytic function in $C_\infty \setminus \Delta_\beta^\alpha$. Since $\varphi(\nu) = 0$ for $\nu \in N \setminus \Lambda$, then f has the form (1).

The Theorem is thus proved.

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Ս. Ե. Մկրտչյան

Լոգարիթմական սեկտորից դուրս հոլոմորֆ, սահմանափակ և լակունար աստիճանային շարքով ներկայացվող ֆունկցիաների միակության մասին

Աշխատանքում ցույց է տրվում, որ դրական ամբողջ թվերի բազմության համար Մյունցի տեսքի պայմանը անհրաժեշտ և բավարար է, որպեսզի գոյություն ունենա լակունար աստիճանային շարք $f(z)=\sum_{\nu\in\Lambda} f_\nu/z^\nu$, որը թույլ է տալիս նախապես տրված փակ լոգարիթմական սեկտորից դուրս անալիտիկ և սահմանափակ շարունակություն:

С. Е. Мкртчян

О единственности голоморфных и ограниченных вне замкнутого логарифмического сектора функций, представимых лакунарными степенными рядами

В настоящей работе показано, что для множества Λ положительных целых чисел условие типа Мюнца необходимо и достаточно для существования лакунарного степенного ряда $f(z)=\sum_{\nu\in\Lambda} f_\nu/z^\nu$, который допускает аналитическое и ограниченное продолжение вне заданного замкнутого логарифмического сектора.