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Mathematics

INITIAL BOUNDARY VALUE PROBLEM FOR SOME CLASS OF NON-LINEAR DEGENERATE PSEUDOPARABOLIC EQUATIONS

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The present paper studies existence and uniqueness of solution of initial boundary value problems for the non-linear degenerate pseudoparabolic equations. It is proved that if operators L and M satisfy certain conditions, the problem has a unique solution in corresponding functional spaces.

***Keywords*:** non-linear operator, pseudoparabolic, monotone operator.

1⁰. Let $\Omega \subset R^n$ be a bounded domain in a half-space $x_n > 0$ with sufficient smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$, where Γ_0 is a domain in hyperspace $x_n = 0$.

The following initial boundary value problem is considered:

$$\frac{\partial}{\partial t} L\big(u(t,x)\big) + M\big(u(t,x)\big) = 0, \tag{1}$$

$$x = (x_1,\ldots,x_n) \in \Omega, \quad t > 0,$$

$$u\big|_{t=0} = u_0(x), \tag{2}$$

$$u\big|_{\Gamma_1} = 0, \tag{3}$$

where operators L and M are defined as

$$L(u) = -\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \frac{\partial}{\partial x_n}\left(b_{nn}(x)\frac{\partial u}{\partial x_n}\right),$$

$$b_{ij}(x) = b_{ji}(x), \quad i,j = 1,2,\ldots,n-1,$$

$$M(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(a_i(x,\nabla u)\big).$$

Assume that functions $b_{ij}(x)$, $i,j = 1,2,\ldots,n-1$, $b_{nn}(x)$, $a_i(x,\vec{\xi})$ are continuous for any $x \in \bar{\Omega}$, $\forall \xi \in R^n$, whereas:

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a) $\left|b_{nn}(x)\right| \le x_n^{\alpha}$, $1<\alpha<2$, and the quadratic form

$$L(x,\xi)=\sum_{i,j=1}^{n-1} b_{ij}(x)\xi_i\xi_j + b_{nn}(x)\xi_n^2$$

is positively defined for all $x\in\Omega$, and for all $x\in\Gamma_0$ its rank is equal to $(n-1)$;

b) $\sum_{i=1}^{n} a_i(x,\xi)\eta_i \le c\left[\sum_{i,j=1}^{n-1} b_{ij}(x)\xi_i\eta_j + b_{nn}(x)\xi_n\eta_n\right]$;

c) $\sum_{i=1}^{n} a_i(x,\xi)\xi_i\xi_j \ge c_1\left(\sum_{i,j=1}^{n-1} b_{ij}(x)\xi_i\xi_j + b_{nn}(x)\xi_n^2\right)$.

For the case $L=\Delta$, $M=\frac{\partial^2}{\partial x^2}$ and $\Gamma_1=\Gamma$ the problem (1)–(3) has been originally investigated by S.L. Sobolev [1], and afterwards similar problems for the case of linear operators have been investigated in works of R.A. Alexandryan, G.V. Virabyan, T.I. Zelenyak and many other armenian and foreign mathematicians [2]. In those works the operator L in equation (1) was considered to be strict elliptic, and M – linear.

In the case of non-linear operators, when $\Gamma_1=\Gamma$ (no degeneracy), the problem (1)–(3) was considered in [3–6]. For operators L, which may be degenerate on the entire boundary, or on a part of it, the problem (1)–(3) was investigated by G.S. Hakobyan [7].

The present paper considers the case of initial boundary value problems (1) for elliptic degenerate L and non-linear M operators.

2^0. Let K_L be a set of functions, which satisfy the following conditions:

1) $u(x)$, $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ $(i,j<n)$, $x_n^{\alpha}\frac{\partial u}{\partial x_n}$ and $\frac{\partial}{\partial x_n}\left(x_n^{\alpha}\frac{\partial u}{\partial x_n}\right)$ are continuous functions in $\bar{\Omega}$,

2) $u(x)$ vanishes in some neighborhood of Γ_1.

In this case the following theorem is true.

Theorem 1 (see [7, 8]). Operator L, defined as a mapping from K_L into space $L_2(\Omega)$, is symmetric and positively defined.

Proof. Let $u(x)$ and $v(x)\in K_L$. Then

$$\left(L(u),v\right)=-\int_{\Omega}\left(\sum_{i,j=1}^{n-1}\frac{\partial}{\partial x_i}\left(b_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\frac{\partial}{\partial x_n}\left(b_{nn}(x)\frac{\partial u}{\partial x_n}\right)\right)v(x)dx=$$

$$=-\int_{\Gamma}\left(\sum_{i,j=1}^{n-1} b_{ij}(x)\cos\widehat{\nu x_i}\cos\widehat{\nu x_j}+b_{nn}(x)\cos^2\widehat{\nu x}\right)\frac{\partial u}{\partial \nu}v\,ds+$$

$$+\int_{\Omega}\left[\sum_{i,j=1}^{n-1} b_{ij}(x)\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}+b_{nn}(x)\frac{\partial u}{\partial x_n}\cdot\frac{\partial v}{\partial x_n}\right]dx=$$

$$=-\int_{\Gamma_0} b_{nn}(x)\frac{\partial u}{\partial x_n}\cdot v\,ds+\int_{\Omega}\left[\sum_{i,j=1}^{n-1} b_{ij}(x)\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}+b_{nn}(x)\frac{\partial u}{\partial x_n}\cdot\frac{\partial v}{\partial x_n}\right]dx\,.$$

We are going to prove that $\int\limits_{\Gamma_0} b_{nn}(x)\frac{\partial u}{\partial x_n} v ds = 0$. To prove this, it's enough to show that

$$\lim_{x_n \to 0} x_n^{\alpha} \frac{\partial u}{\partial x_n} = w(x_1,\ldots,x_n),$$

and the equality $w(\hat{x}) = w(x_1,\ldots,x_n) = 0$ takes place.

The existence of limit above explicitly follows from the definition of set K_L. Assume there exists an element $\hat{x}_0$, for which $w(\hat{x}_0) > 0$. Then, for enough small $x_n > 0$ we get

$$\frac{\partial u(\hat{x}_0, x_n)}{\partial x_n} > \frac{w(\hat{x}_0)}{2x_n^{\alpha}}.$$

Hence, the integral $\int\limits_0^{x_n} \frac{\partial u(\hat{x}_0, x_n)}{\partial x_n} dx_n$ diverges, and we get a contradiction to condition $u(x) \in K_L$.

Thus we proved that $(L(u), v) = (u, L(v))$. Since $L(x,\xi)$ is positive, the operator L is also positive. It is not hard to prove that operator L is positively defined (see [8]).

Let's denote by the same L the Friedrichs extension of operator L, which will be self-adjoint.

3⁰. Let's define a new scalar product on linear manifold K_L by the formula

$$[u,v] = (L(u), v), \tag{4}$$

and let's denote by H_L the closure of manifold K_L by the new norm (derived from the scalar product (4)). So, the functions of H_L have first generalized derivatives by S. L. Sobolev and vanish on boundary Γ_1.

It is known (see [8]) that operator L is invertible and it's inverse operator maps $L_2(\Omega) \to H_L$, hence $\forall u \in K_L$, $L^{-1}Mu \in H_L$.

Thus, we can define an operator A over the domain K_L by the formula

$$A:\ u \to L^{-1}Mu.$$

Now, consider the auxiliary Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = -Au, \\ u\big|_{t=0} = u_0. \end{cases} \tag{*}$$

Lemma 1. The operator $A: H_L : H_L^*$ is bounded.

Proof. Let $u \in K_L$, $v \in K_L$. Then, from definitions of operator A and new scalar product (4) we have

$$\langle Au, v\rangle = \int\limits_{\Omega} Mu \cdot v\, dx = -\sum_{i=1}^{n} \int\limits_{\Omega} \frac{\partial}{\partial x_i} a_i(x, \nabla u)\, v dx =$$

$$=\sum_{i=1}^{n}\int_{\Omega}a_i\left(x,\nabla u\right)\frac{\partial v}{\partial x_i}dx=-\sum_{i=1}^{n}\int_{\Gamma_0}a_i\left(x,\nabla u\right)vds\,.$$

We can show (similary as in Theorem 1) that $\sum_{i=1}^{n}\int_{\Gamma_0}a_i\left(x,\nabla u\right)vds=0$, hence

$$\left\langle Au,v\right\rangle=\sum_{i=1}^{n}\int_{\Omega}a_i\left(x,\nabla u\right)\frac{\partial v}{\partial x_i}dx\,.$$

From the condition (b) it follows that

$$\left\langle Au,v\right\rangle\le c\sum_{i,j=1}^{n-1}\int_{\Omega}\left(b_{ij}\left(x\right)\frac{\partial u}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}+b_{nn}\left(x\right)\frac{\partial u}{\partial x_n}\cdot\frac{\partial v}{\partial x_n}\right)dx=c\left(Lu,v\right)=c\left[u,v\right]\le c\left\|u\right\|_{H_L}\left\|v\right\|_{H_L},$$

therefore $\left\|Au\right\|_{H_L^*}\le c\left\|u\right\|_{H_L}$.

Let X be a reflexive Banach space and X^* is its dual space.

Definition 1. The operator $A:D_A\to X^*$ with the everywhere dense definition domain $D_A\subset X$ is called

1) monotone, if

$$\left\langle Au-Av,u-v\right\rangle\ge 0\quad\forall u,v\in D_A;$$

2) strictly monotone, if there exists such $c_0>0$, so that

$$\left\langle Au-Av,u-v\right\rangle\ge c_0\left\|u-v\right\|^2\quad\forall u,v\in D_A;$$

3) hemicontinuous, if for each $u,v,w\in D_A$ the mapping $\lambda\to\left\langle A\left(u+\lambda v\right),v\right\rangle$ is continuous function from R into R.

It's well known that any strictly monotone operator is coercitive (see [5]), i.e.

$$\exists\alpha>0,\ \left\langle Au,u\right\rangle\ge\alpha\left\|u\right\|^2.$$

Lemma 2. The operator A is hemicontinuous and strictly monotone.

Proof. Suppose $u,v\in K_L$. Then

$$\left\langle Au-Av,u-v\right\rangle=-\sum_{i=1}^{n}\int_{\Omega}\left[\frac{\partial}{\partial x_i}\left(a_i\left(x,\nabla u\right)\right)-\frac{\partial}{\partial x_i}\left(a_i\left(x,\nabla v\right)\right)\right]\left(u-v\right)dx=$$

$$=\sum_{i=1}^{n}\int_{\Omega}\left(a_i\left(x,\nabla u\right)-a_i\left(x,\nabla v\right)\right)\frac{\partial\left(u-v\right)}{\partial x_i}dx\,.$$

From the equation

$$a_i\left(x,\xi\right)-a_i\left(x,\eta\right)=\int_0^1\frac{d}{dt}a_i\left(x,\eta+t\left(\xi-\eta\right)\right)dt=\int_0^1\sum_{j=1}^{n}\frac{da_i\left(x,\eta+t\left(\xi-\eta\right)\right)}{d\xi_i}\left(\xi_j-\eta_j\right)dt$$

it follows that

$$\left\langle Au-Av,u-v\right\rangle=\sum_{i=1}^{n}\int_{\Omega}\int_0^1 a_{ij}\left(x,\nabla v+t\left(\nabla u-\nabla v\right)\right)\frac{\partial\left(u-v\right)}{\partial x_j}\frac{\partial\left(u-v\right)}{\partial x_i}dtdx=$$

$$=\int_0^1\int_{\Omega}\sum_{i=1}^{n}a_{ij}\left(x,\nabla v+t\left(\nabla u-\nabla v\right)\right)\frac{\partial\left(u-v\right)}{\partial x_j}\frac{\partial\left(u-v\right)}{\partial x_i}dxdt\ge$$

$$\geq c_1 \int_\Omega \left[\sum_{i,j=1}^{n-1} b_{ij}(x) \frac{\partial(u-v)}{\partial x_i} \frac{\partial(u-v)}{\partial x_j} + b_{nn}(x) \frac{\partial(u-v)}{\partial x_n} \frac{\partial(u-v)}{\partial x_n} \right] dx = c_1 \|u-v\|_{H_L}^2 .$$

Therefore, the operator A is strictly monotone. The hemicontinuouty of operator A is evident.

Definition 2. The function $u(t,x) \in L_2(0,T;H_L)$ is called be generalized solution to the problem (∗), if $\forall v(t,x) \in L_2(0,T;H_L)$, if the equality

$$\int_0^T \int_\Omega \left[\frac{\partial u}{\partial t} v + \sum_{i=1}^n a_i(x, \nabla u) \frac{\partial v}{\partial x_i} \right] dxdt = 0$$

holds and $u(0,x) = u_0(0)$. The last condition is treated in weak meaning, i.e $\forall w(x) = H_L$,

$$\lim_{t\to 0} \int_\Omega u(t,x) w(x) dx = \int_\Omega u_0(x) w(x) dx .$$

Theorem 2. Let the conditions (a) – (c) be hold. Then $\forall u_0(x) \in H_L$ the problem (*) has a unique solution.

The proof explicitly follows from Lemmas 1 and 2 (see [9]).

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Սկզբնական-եզրային խնդիր վերասերվող ոչ գծային պսևդոպարաբոլային հավասարումների մի դասի համար

Աշխատանքում ուսումնասիրվում է ոչ գծային վերասերվող պսևդոպարաբոլային հավասարումների մի դասի համար սկզբնական-եզրային խնդրի լուծման գոյության և միակության հարցը: Ապացուցվում է, որ եթե L և M դիֆերենցիալ օպերատորները բավարարում են որոշակի պայմանների, ապա խնդիրն ունի միակ լուծում համապատասխան ֆունկցիոնալ տարածությունում:

Начально-краевая задача для одного класса нелинейных вырождающихся псевдопараболических уравнений

В работе рассматривается вопрос о существовании и единственности решения начально-краевой задачи для одного класса нелинейных псевдопараболических уравнений с вырождением.

Доказывается, что если дифференциальные операторы *L* и *M* удовлетворяют некоторым условиям, то начально-краевая задача имеет единственное решение в соответствующем функциональном пространстве.