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Mathematics

AN APPROACH TO INTERPOLATION BY INTEGRATION

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In this paper we consider the correctness of Lagrange bivariate interpolation problem, where interpolation parameters are integrals over certain bounded plane regions. Here we study the case of bivariate polynomials of degree not exceeding one.

***Keywords*:** Lagrange, bivariate, mean-value, interpolation.

Introduction. Denote by Π_k the space of bivariate polynomials of total degree less than or equal k. The classic Lagrange interpolation parameters are the values of a function at given points (x_i, y_i), $i = 1, 2, ..., s$, where s is the number of knots, equals to $\dim \Pi_k$. In other words, the Lagrange interpolation problem is: find a unique polynomial $p \in \Pi_k$ such that $p(x_i, y_i) = c_i$, $i = 1, 2, ..., s$, where c_i are arbitrary predetermined numbers. We consider Lagrange interpolation problem, where interpolation parameters are integrals over certain plane regions. To describe these regions we start with

Definition 1 [1]. We say that a set L of lines in the plane is in general position, if any two lines intersect at one point and no other third line passes through this intersection point.

For a set of lines in general position we call *b*-regions the bounded regions, whose boundary points belong to these lines, while none of their interior point does. In other words, *b*-regions are bounded regions cut by the given set of lines.

For $k+3$ lines in general position, the following lemma is true.

Lemma. Let the lines $L_1, L_2, ..., L_{k+3}$ be in general position, where $k \ge 0$. Then, there are exactly $\binom{k+2}{2}$ *b*-regions.

Proof. We prove it by induction on k. The case $k = 0$ is obvious. Suppose $k \ge 1$, and the formula is true for $k = m$, let us prove it for $k = m+1$. For $m+3$ lines, by assumption, there are $\binom{m+2}{2}$ *b*-regions. Now, let us add one more line in

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the plane, say *l*, to be in general position with the other lines. Thus, there are $m+3$ intersection points on l: $M_1, M_2, ..., M_{m+3}$. Consider the segments M_i, M_{i+1}, $i=1,2,...,m+2$, with neighboring endpoints. Each of them either divides a *b*-region into two bounded regions, whenever it is inside a *b*-region or creates a new *b*-region, otherwise. In each case it makes the number of *b*-region to increase by one. Therefore, the number of *b*-regions increases by $m+2$, which is the number of the segments. Thus, we have $\binom{m+2}{2}+(m+3-1)=\binom{m+3}{2}$ *b*-regions.

Hereon we denote $N:=\binom{k+2}{2}$. We also denote by G_i, $i=1,2,..,N$, the above mentioned *b*-regions.

Now, let us formulate the Lagrange interpolation problem. We are going to find a unique polynomial $p \in \Pi_k$ such that

$$\int\int_{G_i} p(x,y)dxdy = c_i , \quad i=1,2,...,N , \tag{1}$$

where c_i are arbitrary predetermined numbers.

We can express conditions (1) in terms of mean values over *b*-regions: $\frac{1}{\mu(G_i)}\int\int_{G_i} p(x,y)dxdy = c_i'$, $i=1,2,...,N$, where $\mu(G_i)$ is the area of G_i, and $c_i' = \frac{c_i}{\mu(G_i)}$ are arbitrary predetermined numbers. For this reason we refer the considered above interpolation as the *mean value interpolation.*

Definition 2. The mean value Lagrange interpolation problem is said to be correct, if for any given data

$$\Lambda = \{c_i : i=1,2,...,N\} \tag{2}$$

there is a unique polynomial $p \in \Pi_k$ satisfying (1).

The polynomial $p \in \Pi_k$ is of the following form

$$p(x,y) = \sum_{i+j\le k} a_{ij}x^i y^j . \tag{3}$$

Substituting this into (1), we get a linear system of N equations with N unknowns, which are the coefficients of the right-hand side of (3). Therefore, to prove the correctness of the interpolation it is enough to show that the main determinant of the above mentioned linear system is not equal to zero.

The Results. In this section we introduce a hypothesis and prove it in a special case.

Hypothesis. Let the lines $L_1, L_2, ..., L_{k+3}$ be in general position. Then, the mean value interpolation with Π_k is correct.

For the case $k=0$ the hypothesis is true. Indeed, if $p \in \Pi_0$, i.e. *p=const* and $\int\int_G p(x,y)dxdy = 0$, then $p=0$. In this paper we prove that the hypothesis is true also in the case $k=1$.

Theorem. Suppose that the lines L_1, L_2, L_3, L_4 are in general position. Then, the mean-value interpolation with Π_1 is correct.

Proof. We use a linear transformation to make the images of two lines coincide with *X* and *Y* axes. Suppose this linear transformation

$$T : \Pi_1 \to \Pi_1 \tag{4}$$

is given as $X = TU = \begin{pmatrix} f & g \\ h & i \end{pmatrix} \times \begin{pmatrix} u \\ v \end{pmatrix}$. Denote the image of line L_i under transformation T by L_i', $i = 1,...,4.$. We have that L_1' and L_2' are given by equations $u = 0$ and $v = 0$ respectively. Let us denote equations of L_3' and L_4' by $v = \frac{-d}{a}(u - a)$ and $v = \frac{-c}{b}(u - b)$ respectively. Since in this case the Jacobean $J = \frac{\partial(x, y)}{\partial(u, v)}$ is constant, therefore, changing variables, we get the following formula:

$$\int\int_{G_i} p(x, y)dxdy = J\int\int_{\tilde{G}_i} p(fu + gv, hu + iv)dudv. \tag{5}$$

Note that $p(x, y) \in \Pi_1 \Leftrightarrow p(fu + gv, hu + iv) \in \Pi_1$. Therefore, formula (5) implies that the mean-value interpolation with L_1, L_2, L_3, L_4 and Π_1 is correct, if and only if the mean value interpolation with L_1', L_2', L_3', L_4' and Π_1 is correct. Indeed, if for any c_1, c_2, c_3 there exists a unique $p \in \Pi_1$ such that $\int\int_{G_i} p(x, y)dxdy = c_i, i = 1,2,3$, then for any c_1', c_2', c_3' there exists a unique $\tilde{p} \in \Pi_1$ such that $\int\int_{\tilde{G}_i} \tilde{p}(u, v)dudv = c_i',\ i = 1,2,3.$

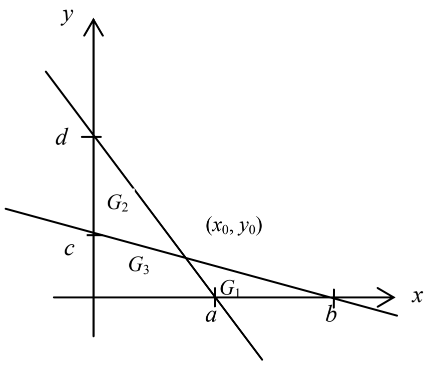
Hence, it is enough to prove the correctness in case of L_1', L_2', L_3', L_4'. Here we replace (u, v) by (x, y). We shall prove the correctness of the interpolation by calculating its *Vandermonde* determinant:

$$\det A = \begin{vmatrix} \int\int_{G_1} 1dxdy & \int\int_{G_2} 1dxdy & \int\int_{G_3} 1dxdy \\ \int\int_{G_1} xdxdy & \int\int_{G_2} xdxdy & \int\int_{G_3} xdxdy \\ \int\int_{G_1} ydxdy & \int\int_{G_2} ydxdy & \int\int_{G_3} ydxdy \end{vmatrix}, \tag{6}$$

and showing that it is not zero. Let us start with calculating the elements' of the determinant.

In view of elementary operations of determinant we can replace the integrals over G_1 and G_2 in (6) by the integrals over two big triangles $G_1' := G_1 \cup G_3$ and $G_2' := G_2 \cup G_3$ respectively (see Figure). Let (x_0, y_0) be the intersection of L_3' and L_4'. Taking into consideration their equations, we have

$$(x_0, y_0) = \left(\frac{(c - d)ab}{ac - bd}, \frac{(a - b)cd}{ac - bd} \right). \tag{7}$$



b-regions.

Now we get by simple calculations:

$$\iint_{G_1'} 1dxdy = \frac{1}{2}bc, \qquad \iint_{G_2'} 1dxdy = \frac{1}{2}ad,$$

$$\iint_{G_3'} 1dxdy = \frac{1}{2}(x_0+a)y_0 + \frac{1}{2}(c-y_0)x_0 = \frac{acb(c-d)(-2bd+a(c+d)}{2(ac-bd)^2} + \frac{a(a-b)^2c^2d}{2(ac-bd)^2} =$$

$$= \frac{ac(b(c-2d)+ad)}{2(ac-bd)},$$

$$\iint_{G_1'} xdxdy = \int_0^b \int_0^{-\frac{c}{b}(x-b)} xdydx = \frac{1}{6}cb^2, \qquad \iint_{G_1'} xdxdy = \int_0^b \int_0^{-\frac{c}{b}(x-b)} ydydx = \frac{1}{6}c^2b,$$

$$\iint_{G_2'} xdxdy = \int_0^a \int_0^{\frac{-d}{a}(x-a)} xdydx = \frac{1}{6}a^2d, \qquad \iint_{G_2'} ydxdy = \int_0^a \int_0^{\frac{-d}{a}(x-a)} ydydx = \frac{1}{6}ad^2.$$

Now it remains to calculate the integrals of x and y over G_3. We have

$$\iint_{G_3} xdxdy = \int_0^{x_0} \int_0^{\frac{-c}{b}(x-b)} xdydx + \int_{x_0}^{a} \int_0^{\frac{-d}{a}(x-a)} xdydx = \frac{a^2c(a^2cd-2abd^2+b^2(c^2-3cd+3d^2))}{6(ac-bd)^2}.$$

In the similar way we get

$$\iint_{G_3} ydxdy = \frac{ac^2(a^2d^2+b^2d(-2c+3d)+ab(c^2-3d^2))}{6(ac-bd)^2}.$$

Now we get for the Vandermonde determinant

$$\det A = \begin{vmatrix} \frac{1}{2}bc & \frac{1}{2}ad & \frac{ac(b(c-2d)+ad)}{2(ac-bd)} \\ \frac{1}{6}cb^2 & \frac{1}{6}da^2 & \frac{a^2c(a^2cd-2abd^2+b^2(c^2-3cd+3d^2))}{6(ac-bd)^2} \\ \frac{1}{6}bc^2 & \frac{1}{6}ad^2 & \frac{ac^2(a^2d^2+b^2d(-2c+3d)+ab(c^2-3d^2))}{6(ac-bd)^2} \end{vmatrix}.$$

Now using elementary operations of determinant, we obtain

$$\det A = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{1}{6} bcad \begin{vmatrix} 1 & 1 & \dfrac{ac(b(c-2d)+ad)}{ac-bd} \\ b & a & \dfrac{a^2c(a^2cd-2abd^2+b^2(c^2-3cd+3d^2))}{(ac-bd)^2} \\ c & d & \dfrac{ac^2(a^2d^2+b^2d(-2c+3d)+ab(c^2-3d^2))}{(ac-bd)^2} \end{vmatrix} =$$

$$= \frac{abcd}{72(ac-bd)^2} \begin{vmatrix} 1 & 1 & ac(b(c-2d)+ad)(ac-bd) \\ b & a & a^2c(a^2cd-2abd^2+b^2(c^2-3cd+3d^2)) \\ c & d & ac^2(a^2d^2+b^2d(-2c+3d)+ab(c^2-3d^2)) \end{vmatrix} =$$

$$= \frac{a^2bc^2d}{72(ac-bd)^2} \begin{vmatrix} 1 & 1 & (b(c-2d)+ad)(ac-bd) \\ b & a & a(a^2cd-2abd^2+b^2(c^2-3cd+3d^2)) \\ c & d & c(a^2d^2+b^2d(-2c+3d)+ab(c^2-3d^2)) \end{vmatrix}.$$

Finally, we obtain

$$\det A = \frac{(abcd)^2(a-b)^2(c-d)^2}{36(ac-bd)^2} . \tag{8}$$

Note that $ac \neq bd$ since L_3' and L_4' are not parallel. It is also easy to see that the numerator in the right-hand side of (8) is not zero, since L_1', L_2', L_3', L_4' are in the general position. Thus $\det A \neq 0$. Therefore, the mean value interpolation with Π_1 is correct.

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Մի մոտեցում միջարկմանը ինտեգրումով

Հոդվածում քննարկվում է Լագրանժի երկչափ միջարկման (ինտերպոլյացիայի) խնդրի ճշգրտությունը, երբ միջարկային պարամետրերը ինտեգրալներ են որոշ սահմանափակ հարթ տիրույթներով: Քննարկվում է մեկը չգերազանցող աստիճան-ներով երկչափ բազմանդամների դեպքը:

Один подход к интерполяции, основанный на интегрировании

В статье рассматривается корректность двумерной интерполяционной задачи Лагранжа, где интерполяционные параметры – интегралы по некоторым ограниченным плоским областям. Изучен случай двумерных многочленов степени, не превосходящей единицу.