PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2009, № 2, p. 26–31

Mathematics

BIVARIATE INTERPOLATION WITH INTEGRALS

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The bivariate interpolation problem, where the interpolation parameters are integrals over bounded regions, is considered in this paper. H. Hakopian posed a hypothesis for this problem in the case, when regions are obtained from intersection of lines in general position [2]. Till now the hypothesis is proved for polynomials of degree ≤ 1. In this paper we bring a new proof. Meanwhile we solve the problem in more general setting – in the case of arbitrary regions.

***Keywords*:** correctness, bivariate, centroid, interpolation.

Denote by Π_n the space of bivariate algebraic polynomials of total degree not exceeding n: $\Pi_n = \{p(x,y) = \sum_{i+j\leq n} a_{ij}x^i y^j, \quad a_{ij} \in R\}$. Denote

$$N \overset{def}{=} \dim \Pi_n = \binom{n+2}{2}.$$

Let us fix the set of points $\mathscr{X}_s = \{(x_1,y_1),(x_2,y_2),...,(x_s,y_s)\} \subset R^2$ as the set of nodes of interpolation. The interpolation problem $(\Pi_n, \mathscr{X}_s)$ is called correct, if for any values $\{c_1,c_2,...,c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions $p(x_k,y_k) = c_k, \quad k = 1,...,s.$ Note that those conditions may be reduced to a system of s linear equations with N unknowns:

$$p(x_k,y_k) = \sum_{i+j\leq n} a_{ij}x_k^i y_k^j = c_k, \quad k = 1,...,s, \tag{1}$$

where a_{ij} are unknowns.

Here the correctness of interpolation means that the linear system has a unique solution for any right side values $\{c_1,c_2,...,c_s\}$. From here we get the necessary correctness condition: $s = N$. It means that the number of unknowns is equal to the number of equations, and hereon we assume that this condition holds. In this case the linear system has a unique solution for any values $\{c_1,c_2,...,c_s\}$, if and only if the corresponding homogeneous system has only zero solution. In other words:

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Statement 1. The interpolation problem $(\Pi_n, \mathscr{X}_N)$ with the node set

$$\mathscr{X}_N = \{(x_j, y_j)\}_{j=1}^N \subset R^2, \quad N = \binom{n+2}{2},$$

is correct, if and only if $p \in \Pi_n$ and $p(x_j, y_j) = 0, \quad j = 1,\dots,N \quad \Rightarrow \quad p \equiv 0.$

Later we'll use this statement in the case of $n = 1$, i.e. the problem with a node set $\mathscr{X}_3 = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, and Π_1 is correct, if and only if $p \in \Pi_1$ and $p(x_j, y_j) = 0, \quad j = 1,2,3 \quad \Rightarrow \quad p \equiv 0$. It means that three points are correct for Π_1, if and only if they are not collinear.

We'll consider interpolation problem, where the interpolation parameters are integrals over bounded regions. Assume that $\mathscr{D}_s = \{D_1, D_2, \dots, D_s\}$ is a set of bounded regions.

Definition. We say that the interpolation problem $(\Pi_n, \mathscr{D}_s)$ is correct, if for any values $\{c_1, c_2, \dots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$, satisfying the conditions

$$\iint_{D_k} p(x,y)dxdy = c_k, \quad k = 1,\dots,s.$$

The considerations and statement carried out in the case of pointwise interpolation hold also in this case, because this problem is reducing to a system of linear equations too:

$$\iint_{D_k} p(x,y)dxdy = \iint_{D_k} \sum_{i+j\le n} a_{ij} x^i y^j dxdy = \sum_{i+j\le n} a_{ij} \iint_{D_k} x^i y^j dxdy = c_k, \quad k = 1,\dots,s.$$

Thus, we have

Statement 2. The problem with the set of regions $\mathscr{D}_3 = \{D_1, D_2, D_3\}$ and Π_1 is correct, if and only if

$$p \in \Pi_1 \text{ and } \iint_{D_j} p(x,y)dxdy = 0, \quad j = 1,2,3 \quad \Rightarrow \quad p \equiv 0. \tag{2}$$

Definition (see [1], chapter 13). We say that the distinct lines $\ell_1, \ell_2, \dots, \ell_s$ are in general position, if any two of them are not parallel, and any three are not passing through a point.

Generally the following problem is considered in the paper.

Suppose $n+3$ distinct lines ($n \ge 0$ is an integer) are given in the plane, which are in general position. It is known that $\binom{n+2}{2}$ bounded regions arise as a result of the line intersection (see [2]). Denote these regions by Δ_i, $i = 1,\dots,N$, and set

$$\mathscr{A}_N = \{\Delta_1, \Delta_2, \dots, \Delta_N\}.$$

See Fig. 1 for the case $n = 2$.

H. Hakopian posed [2] the following

Hypothesis. For any numbers $c_1, c_2, ..., c_N$ there exists a unique polynomial $p \in \Pi_n$ such that $\iint\limits_{\Delta_i} p(x,y)dxdy = c_i$, $i = 1,...,N$. Thus, the problem $(\Pi_n, Д_N)$ is correct.

Till now Hypothesis is proved for the polynomials of degree ≤ 1 (see [2]). The statement is obvious for the polynomials of degree zero, since if we have three lines, which form a bounded region, then by applying Statement 2 we get

$\iint\limits_{\Delta} p_0 dxdy = 0, \quad p_0 = const \quad \Rightarrow \quad p_0 \equiv 0.$

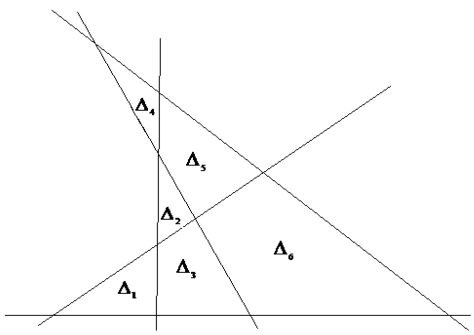

Fig. 1.

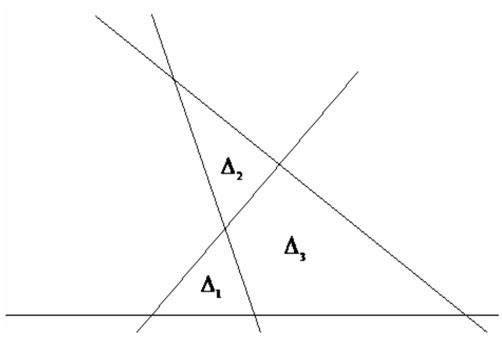

Fig. 2.

Consider the case $n = 1$. Four lines are given, which are in general position. Note that all the bounded regions arising as a result of the intersection are lying inside an angle, formed by a pair of lines. As a result we get two triangles and a tetragon. Denote the triangles by Δ_1, Δ_2 and the tetragon by Δ_3 (see Fig. 2).

In this case the validity of Hypothesis is proved in [2] by evaluation of the Vandermonde determinant. Below we bring a new proof, moreover, the problem is solved in more general setting. According to Statement 2, it suffices to prove that

$$p \in \Pi_1 \text{ and } \iint\limits_{\Delta_i} p(x,y)dxdy = 0, \quad i = 1,2,3 \quad \Rightarrow \quad p \equiv 0.$$

From now on let us denote by ℓ the both line and the polynomial $\ell \in \Pi_1$, which takes part in the equation of the line $\ell(x,y) = 0$. We start by clarifying a problem.

A bounded region with nonzero area is given in the plane. What condition is to be set on a polynomial of degree 1, i.e. a line, such that its integral over the region becomes zero. It turns out that the answer is related with the centroid of the region.

Definition. The centroid of the region D is the point with the coordinates $x^* = S_D^{-1} \iint\limits_D xdxdy$, $y^* = S_D^{-1} \iint\limits_D ydxdy$, where S_D is the area of the region D.

We get immediately from here that

$$\iint\limits_D (Ax + By + C)dxdy = (Ax^* + By^* + C)S_D \, . \tag{3}$$

The next statement follows from (3).

Statement 3. If D is a bounded region with nonzero area and $\ell(x,y)=Ax+By+C$ is a linear polynomial, then $\iint_D \ell(x,y)dxdy=0$, if and only if the line ℓ passes through the centroid of D.

Now consider an interpolation problem with Π_1 and any bounded regions $\mathscr{D}_3=\{D_1,D_2,D_3\}$ of nonzero areas.

Below it is proved that the interpolation problem $(\Pi_1,\mathscr{D}_3)$ is correct, if and only if the pointwise interpolation problem with the centroids of D_1,D_2,D_3 is correct.

Theorem 1. The interpolation problem $(\Pi_1,\mathscr{D}_3)$ is correct, if and only if the centroids of D_1,D_2,D_3 are not collinear.

Proof. Let the centroids of the regions D_1,D_2,D_3: $(x_1^*,y_1^*),(x_2^*,y_2^*),(x_3^*,y_3^*)$ belong to a line ℓ. Then according to Statement 3 we have that

$$\iint_{D_i} \ell(x,y)dxdy=0,\quad i=1,2,3. \tag{4}$$

Since $\ell\neq 0$, then by Statement 2 the interpolation problem is not correct.

Now assume that the centroids of the regions are not collinear. Then it follows from (4) that $\ell\left(x_i^*,y_i^*\right)=0,\quad i=1,2,3.$ We get from here that $\ell(x,y)\equiv 0$ and, therefore, the interpolation problem is correct, according to Statement 2. Using the well-known fact that the centroid of triangle is the point of intersection of medians, we get from Theorem 1

Corrolary. If we have any three triangles in the plane, then the interpolation problem with integrals over these triangles is correct, if and only if the intersection points of medians of the triangles are not collinear. Now let us return to the result proved in [2].

Theorem 2 [2]. Let 4 lines are given, which are in general position, and the bounded regions Δ_i, $i=1,2,3$, arise as a result of intersection. Then the interpolation problem $\left(\Pi_1,\mathscr{A}_3\right)$, where $\mathscr{A}_3=\{\Delta_1,\Delta_2,\Delta_3\}$ is correct.

Let us bring a new proof of Theorem 2, which is based on Theorem 1.

Let us start with the following

Lemma. Suppose that $p(x,y)=Ax+By+C$ is a linear polynomial, and Δ is a triangle with vertices $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) with the area S_Δ. Then

$$\iint_\Delta p(x,y)dxdy=\frac{S_\Delta}{3}\left[p(M_1)+p(M_2)+p(M_3)\right],$$

where the points M_1,M_2,M_3 are the midpoints of sides of the triangle Δ.

Proof. Applying (3), we get

$$\iint_\Delta (Ax+By+C)dxdy=S_\Delta(A\frac{x_1+x_2+x_3}{3}+B\frac{y_1+y_2+y_3}{3}+C)=$$

$$=\frac{S_\Delta}{3}(A(\frac{x_1+x_2}{2}+\frac{x_2+x_3}{2}+\frac{x_3+x_1}{2})+B(\frac{y_1+y_2}{2}+\frac{y_2+y_3}{2}+\frac{y_3+y_1}{2})+3C)=$$

$$=\frac{S_\Delta}{3}\Big(A\frac{x_1+x_2}{2}+B\frac{y_1+y_2}{2}+C+A\frac{x_2+x_3}{2}+B\frac{y_2+y_3}{2}+C+$$

$$+A\frac{x_3+x_1}{2}+B\frac{y_3+y_1}{2}+C\Big).$$

This completes the proof.

The proof of Theorem 2. According to Statement 2, it suffices to verify that

$$p\in\Pi_1 \ \text{ and } \ \iint_{\Delta_i} p(x,y)dxdy=0,\quad i=1,2,3 \quad\Rightarrow\quad p\equiv 0.$$

Suppose conversely that there is $\ell\in\Pi_1$, $\ell\neq 0$ such that $\iint_{\Delta_i}\ell(x,y)dxdy=0$, $i=1,2,3$.

From here by Statement 3 the line ℓ passes through the centroids of the triangles $(\Delta_1)=(EAB)$ and $(\Delta_2)=(BCF)$ (see Fig. 3).

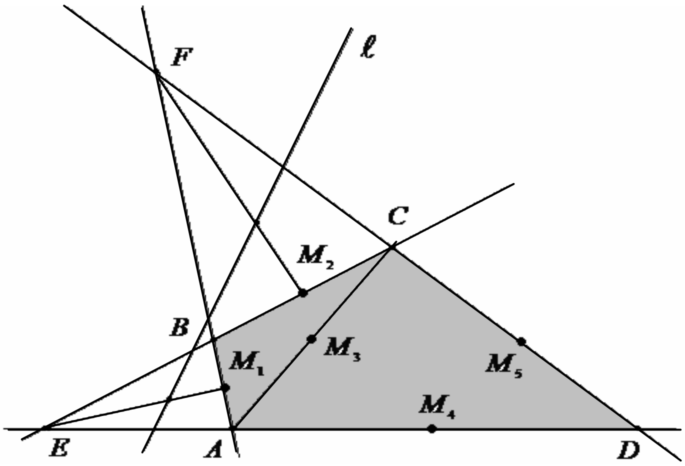

Fig. 3.

For contradiction we'll show that $\iint_{\Delta_3}\ell(x,y)dxdy\neq 0$. We have

$$\iint_{\Delta_3}\ell(x,y)dxdy=\iint_{(ABC)}\ell(x,y)dxdy+\iint_{(ACD)}\ell(x,y)dxdy.$$

To calculate the integrals over the triangles (ABC) and (ACD) let us apply the formula of Lemma. We'll show that the midpoints of all the sides of the triangles are lying on the same side of the line ℓ (see Fig. 3). Indeed, the midpoints of the sides AB and BC of the triangle (ABC) are lying on the same side of ℓ, since they are on the continuation of medians from the vertices E and F of the triangles Δ_1 and Δ_2. Note that if we choose arbitrarily one by one points in the triangles (EAB) and (BCF) except the points A and C, and connect them by line, then the vertices A and C will be on the same side of that line. Therefore, the same thing occurs in the case of the line between the centroids of the triangles.

That is the vertices A and C as well as the midpoint of AC are lying on the same side of ℓ. Lastly, the midpoints of AD and CD are on the same side of ℓ since the vertices A, D, and C are on the same side. Therefore, applying Lemma for the triangles (ABC) and (ACD), we get

$$\iint_{\Delta_3} \ell(x,y)dxdy == \frac{S_{ABC}}{3}\left[\ell(M_1)+\ell(M_2)+\ell(M_3)\right]+\frac{S_{ACD}}{3}\left[\ell(M_3)+\ell(M_4)+\ell(M_5)\right].$$

The obtained expression can not be zero, since due to above considerations the values of $\ell(x,y)$ at all those midpoints have the same sign. This completes the proof.

We specially thank to Professors J. Avetisyan and H. Hakopian for the useful considerations and advices concerning the posed problem.

Received 28.12.2008

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Երկչափ միջարկում ինտեգրալներով

Աշխատանքում ուսումնասիրվում է երկչափ միջարկման խնդիր, որտեղ մի-ջարկման պարամետրերը ինտեգրալներ են սահմանափակ տիրույթներով: Հ. Հակոբյանի կողմից [2] առաջադրված է վարկած այդ խնդրի համար այն դեպքում, երբ տիրույթներն առաջանում են ընդհանուր դրության մեջ գտնվող ուղիղների հատման արդյունքում: Վարկածը մինչ այժմ ապացուցված է աստիճանի բազմանդամների համար: Աշխատանքում բերված է նոր ապացույց, ընդ որում նախապես խնդիրը լուծ-ված է ավելի ընդհանուր դրվածքով՝ ցանկացած սահմանափակ տիրույթների դեպքում:

Двумерная интерполяция с интегралами

В работе изучена двумерная интерполяционная задача, в которой интерполя-ционные параметры – интегралы по ограниченным областям. А. Акопяном [2] предложена гипотеза для этой задачи в случае, когда области получаются в результате пересечения прямых, находящихся в общем положении. До сих пор гипотеза доказана для полиномов степени ≤ 1. В этой работе приведено новое доказательство, при этом задача заранее решена в более общей постановке – в случае произвольных ограниченных областей.