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Mathematics

INFINITE ORDER AUTOMORPHISMS OF FREE PERIODIC GROUPS OF SUFFICIENTLY LARGE EXPONENT

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In this paper we construct infinite order automorphisms of free periodic groups $B(m,n)$ of sufficiently large period n with $m \geq 2$ generators. From the obtained results it follows that the quotient group of the group $\mathrm{Aut}(B(m,n))$ with respect to normal subgroup of inner automorphisms is infinite.

***Keywords*:** free periodic groups, Burnside groups, group automorphisms.

Introduction. "Let n be a sufficiently large odd number. Characterize automorphisms of a free Burnside group $B(m,n)$ of period n with m generators". This is the problem № 8.53, a), proposed by A.Yu. Olshanskii in [1] in 1982. Recently E.A. Cherepanov published two works [2, 3] devoted to the study of automorphism groups of free Burnside groups $B(m,n)$. Namely, in the work [2] normal automorphisms of groups $B(m,n)$ for odd $n > 10^{10}$ and $m \geq 2$ are described. In [3] the existence of a free subsemigroup in the group $\mathrm{Aut}(B(m,n))$ was shown.

We construct new automorphisms of infinite order in the group $\mathrm{Aut}(B(m,n))$.

The following notations are used: $\overline{X}$ is a word X written on a circle, $U \equiv V$ means letter-for-letter equality of words U and V. $\partial(X)$ is the length of word X.

Let us consider the free Burnside group $B(2,n)$ with basis $\{a,b\}$ and two automorphisms, given as follows:

$$\varphi : a \mapsto b, \quad \varphi : b \mapsto a^2 b, \tag{1}$$

$$\psi : a \mapsto ab^2, \ \psi : b \mapsto ab^3. \tag{2}$$

Our main purpose is to prove

Theorem 1. Suppose that $n > 10^{10}$ is an arbitrary odd number. Then the automorphism φ of group $B(2,n)$, defined by relation (1), has infinite order in $\mathrm{Aut}(B(2,n))$.

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Theorem 2. Suppose that $n > 10^{10}$ is either an arbitrary odd, or $n = 16k \geq 8000$ is an arbitrary even number. Then the automorphism ψ of group $B(2,n)$, defined by relation (2), has infinite order in $\mathrm{Aut}(B(2,n))$.

Since any inner automorphism of the group $B(2,n)$ has a finite order, from Theorems 1, 2 immediately implies:

Corollary 1. For any natural k automorphisms φ^k and ψ^k, defined by relations (1) and (2), are not inner.

Corollary 2. The quotient group $\mathrm{Aut}(B(2,n))/\mathrm{Int}(B(2,n))$ is infinite, where $\mathrm{Int}(B(2,n))$ is the group of inner automorphisms of group $B(2,n)$.

From Theorems 1, 2 similar statements for $\mathrm{Aut}(B(m,n))$ for $m \geq 3$ immediately follow. We rather define automorphisms $\bar{\varphi}$ and $\bar{\psi}$ of the group $B(m,n)$ as follows: $\bar{\varphi}(a_1) \rightleftharpoons \varphi(a_1)$, $\bar{\varphi}(a_2) \rightleftharpoons \varphi(a_2)$, $\bar{\varphi}(a_i) \rightleftharpoons id(a_i)$, $3 \leq i \leq m$, $\bar{\psi}(a_1) \rightleftharpoons \psi(a_1)$, $\bar{\psi}(a_2) \rightleftharpoons \psi(a_2)$, $\bar{\psi}(a_i) \rightleftharpoons id(a_i)$, $3 \leq i \leq m$. Then holds

Corollary 3. For $m \geq 3$ and arbitrary large natural n automorphisms $\bar{\varphi}$ and $\bar{\psi}$ have infinite order in $\mathrm{Aut}(B(m,n))$, and the quotient group $\mathrm{Aut}(B(m,n))/\mathrm{Int}(B(m,n))$ is infinite.

1. Auxiliary lemmas for automorphism φ.

Let $X_l \rightleftharpoons \varphi^l(a)$, $Y_l \rightleftharpoons \varphi^l(b)$ for any $l \geq 1$.

Lemma 1.1. For any $l \geq 1$ the following relations hold:

a) $X_{l+1} \equiv Y_l \equiv X_{l-1}^2 X_l$;

b) $\partial(X_l) = \dfrac{2^{l+1} + (-1)^l}{3}$;

c) $\begin{cases} 2\partial(X_l) = \partial(X_{l+1}) - 1 & \text{for odd } l; \\ 2\partial(X_l) = \partial(X_{l+1}) + 1 & \text{for even } l; \end{cases}$

d) $17\partial(X_{l-4}) \geq \partial(X_l)$ for $l \geq 6$;

e) $\partial(X_{l-1}) + 1 < 0.51\partial(X_l)$ for $l \geq 8$;

f) $9\partial(X_{l-3}) > \partial(X_l)$ for $l \geq 6$.

Proof. a) $X_{l+1} \equiv \varphi^{l+1}(a) \equiv \varphi^l(b) \equiv Y_l$. The second equality follows from $X_2 \equiv X_0^2 X_1$ by induction, where $X_0 \rightleftharpoons a$.

b) $\partial(X_1) = 1$, $\partial(X_2) = 3$. For any natural l denote

$$\partial(X_l) = x_l,\ a_l = \partial(X_l) + \partial(X_{l-1}) = x_l + x_{l-1},\ b_l = x_l - 2x_{l-1}.$$

We have $x_{l+1} + x_l = 2(x_l + x_{l-1})$ and $x_{l+1} - 2x_l = -(x_l - 2x_{l-1})$. From here we obtain the equalities $x_l + x_{l-1} = 2^l$, $x_l - 2x_{l-1} = (-1)^l$ and, finally, $x_l = \dfrac{2^{l+1} + (-1)^l}{3}$.

Proofs of statements c, d, e and f immediately follow from statement b.

Lemma 1.2. If $A^t A' \equiv B^r B'$, where the word A' is a start of A, B' is a start of B and $\partial(A^t A') \geq \partial(AB)$, there exists a word D such that $A \equiv D^k$ and $B \equiv D^s$ for some k and s. In particular, if A is a simple word, then $A \equiv D$ and $B \equiv A^s$.

For proof see item 1.2.9 of monograph [4].

Denote $X_{l-1} \rightleftharpoons A$, $X_{l-2} \rightleftharpoons B$, $X_{l-3} \rightleftharpoons C$, $X_{l-4} \rightleftharpoons D$. By Lemma 1.1 we have $X_l \equiv X_{l-2}^2 X_{l-1} \equiv BBA$. Note that A ends with B.

Later on, we suppose that Z is a simple word.

Lemma 1.3. If the word X_l is not a proper power, i.e. if for some word Z and integer k the equality $X_l \equiv Z^k$ holds, then $k = 1$.

Proof. The proof is carried out by induction on l.

1. Since $\partial(X_l)$ is odd, the case $k = 2$ is impossible. Thus $k \geq 3$ and k is odd.

2. If $X_l \equiv Z^3 \equiv BBA$, then $\partial(B) < \partial(Z) < \partial(A)$. Therefore, Z starts with B, and Z is an end of A. Since A ends with word B and $\partial(B) < \partial(Z)$, then B is an end of Z. Hence, either $Z \equiv BFB$ or $Z \equiv B_1 F B_2$, where $B_1 F \equiv F B_2 \equiv B$. In the case of $Z \equiv BFB$ we have $BFBBFBBFB \equiv BBA$ and $\partial(A) = 4\partial(B) + 3\partial(F)$. This contradicts the inequality $\partial(A) \leq 2\partial(B) + 1$. Thus, $Z \equiv B_1 F B_2$ and $\partial(B_1) = \partial(B_2)$, then $B_2 B_1 F B_2 B_1 \equiv B_1 F X_{l-3}^2$. Therefore, $B_1 \equiv B_2$ and, finally, $B \equiv B_1 F \equiv F B_1$. It means (see [4], point I.4.2) that $B_1 \equiv G^k, F \equiv G^s$ and $B \equiv G^{k+s}$.

3. Suppose that $X_l \equiv Z^k$, $k \geq 5$. Since $l \geq 5$, from $k\partial(Z) = 4\partial(B) \pm 1$ we obtain $\partial(Z) < \partial(B)$. Therefore $BB \equiv Z^t Z_1$, where $t \geq 2$, and by Lemma 1.2 we obtain $B \equiv Z^k$. It contradicts the inductive assumption.

Remark 1.1. If $l \leq 5$, then the lengths of words X_1, X_2, X_3, X_4 are less than 17 and the word $\overline{X_5} \equiv \overline{bbaabbbaabaabaabbbaab}$ obviously doesn't contain a subword of form Z^4.

Lemma 1.4. If $\overline{X_l}$ contains Z^k, then $k < 17$.

Proof. The proof is carried out by induction on l. For $l \leq 5$ the validity of the statement follows from Remark 1.1. Assume that $\overline{X_l} \equiv \overline{BBA} \equiv \overline{DDCDDCCCDDC}$ contains Z^k and $k \geq 17$. Since $l \geq 6$, then $\partial(BBA) \leq 17\partial(D)$, and we obtain $\partial(Z) < \partial(D)$. Indeed, assuming that $\partial(Z) \geq \partial(D)$, we obtain $\partial(BBA) = \partial(Z^{17})$ and, therefore, $\overline{BBA} = Z^{17}$. It contradicts Lemma 1.3.

Suppose that the length of the minimal subword of word $\overline{DDCDDCCCDDC}$ over the alphabet $\{C, D\}$ that covers the word Z^k is equal to 5. Taking into consideration that C ends with D, by direct checking we make sure that either C^2 or D^2 occurs in Z^k. Then by Lemma 1.2 either C or D is a proper power in defiance of Lemma 1.3.

Let us show that there doesn't exist a subword of word $\overline{DDCDDCCCDDC}$ over the alphabet $\{C, D\}$ with length 3 that covers Z^k. Indeed, otherwise it would occur in cyclic word $\overline{A}$, that contradicts the inductive assumption.

It remains to consider the case when the length of minimal subword M of word $\overline{DDCDDCCCDDC}$ over the alphabet $\{C, D\}$ that covers the word Z^k is equal to 4. If M letter-for-letter coincides with one of the words $CDCC$, $DDCC$, $DCCC$, $CDDC$, $CCCD$ or $CCDD$, then Z^k occurs in $\overline{A}$, that contradicts the inductive assumption. But if $M \equiv DCDD$, then D^2 occurs in Z^k as an end of the base of occurrence $D * CD * D$, and by Lemma 1.2 D is a proper power. It contradicts Lemma 1.3. It remains to consider the case $M \equiv DDCD$. By definition we have $D \equiv X_{l-4} \equiv X_{l-6}X_{l-6}X_{l-5}$, $C \equiv X_{l-3} \equiv X_{l-5}X_{l-5}X_{l-4}$. Since $l \geq 6$, then from statement $c)$ of Lemma 1.1 the inequality $\partial(DDCD) < 15\partial(X_{l-5})$ follows. It means that $\partial(Z) < \partial(X_{l-5})$ and X^{3}_{l-5} occurs in Z^k. By Lemma 1.2 the word X_{l-5} is a proper power. It contradicts Lemma 1.3. The Lemma is thus proved.

2. Auxiliary lemmas for automorphism ψ.

Now we consider the case of automorphism ψ; $\psi : a \mapsto ab^2$, $\psi : b \mapsto ab^3$. Let $U_l \rightleftharpoons \psi^l(a)$, $V_l \rightleftharpoons \psi^l(b)$ for any $l \geq 1$. We have $U_{l+1} \equiv \psi(U_{l-1})\psi(V_{l-1})^2$ and $V_{l+1} \equiv \psi(U_{l-1})\psi(V_{l-1})^3$. Therefore $U_l \equiv U_{l-1}V_{l-1}^2$ and $V_l \equiv U_{l-1}V_{l-1}^3$. Since the inequality $3\partial(U_t) + 6\partial(V_t) > 2\partial(U_t) + 6\partial(V_t)$ holds for any $t \geq 0$, then $3\partial(U_tV_tV_t) > 2\partial(U_tV_tV_tV_t)$ and, particularly, $3\partial(U_{l-1}V_{l-1}V_{l-1}) > 2\partial(U_{l-1}V_{l-1}V_{l-1}V_{l-1})$. Denote $U_{l-1} = B$, $V_{l-1} = A$, $U_{l-2} = D$, $V_{l-2} = C$. Then, obviously, $3\partial(B) > 2\partial(A)$ and $3\partial(D) > 2\partial(C)$ hold.

Lemma 2.1. U_l and V_l are not proper powers.

Proof. The proof is carried out by induction on l. For $l \leq 2$ the statement is obvious since $U_1 \equiv abb$, $V_1 \equiv abbb$, $U_2 \equiv abbabbbabbb$, $V_1 \equiv abbabbbabbbabbb$. Thus $l \geq 3$.

1. Let $U_l \equiv Z^k$ and $k \geq 2$. Then $BAA \equiv DCCDCCCDCCC \equiv \underbrace{ZZ...Z}_{k}$. Recording the words of this equality in a reverse order and shifting the last D to start, we obtain $\underbrace{Z_1Z_1...Z_1}_{k} \equiv D_1C_1C_1C_1D_1C_1C_1C_1D_1C_1C_1$ and $(D_1C_1C_1C_1)^2D_1C_1C_1 \equiv Z_1^k$. Since $\partial(D_1C_1C_1C_1D_1C_1C_1) > \partial(Z_1)$, then by Lemma 1.2 we have $D_1C_1C_1C_1 \equiv Z_1{}^p$, where $p \geq 2$, because $p \neq 1$ by $\partial(D_1C_1C_1) < \partial(Z_1)$. Hereof it immediately follows that A is a proper power. It contradicts the inductive assumption.

2. Now suppose that $V_l \equiv Z^k$ and $k \geq 2$. Then $V_l \equiv BAAA \equiv DCCDCCCDCCCDCCC \equiv Z^k$. As in item 1 we have $\underbrace{Z_1Z_1...Z_1}_{k} \equiv D_1C_1C_1C_1D_1C_1C_1C_1D_1C_1C_1C_1D_1C_1C_1$, $(D_1C_1C_1C_1)^3D_1C_1C_1 \equiv Z_1^k$,

$\partial(D_1C_1C_1C_1D_1C_1C_1C_1D_1C_1C_1) > \partial(Z_1)$, and again by Lemma 1.2 A is a proper power. The obtained contradiction proves Lemma 2.1.

Lemma 2.2. If $\overline{U_l}$ or $\overline{V_l}$ contains Z^k, then $k \le 20$.

Proof. The proof is carried out by induction on l. For $l \le 2$ the statement follows from lengths of $\overline{U_l}$ and $\overline{V_l}$. Suppose $l \ge 3$ and $k \ge 21$. We have $U_l \equiv BAA$, $V_l \equiv BAAA$, $B \equiv DCC$, $A \equiv DCCC$. Note that Z^k can't occur in $\overline{CDCC}$ or in $\overline{CDC}$, since ,otherwise, it would occur in $\overline{A}$ or $\overline{B}$, that contradicts the inductive assumption. As far as $21\partial(Z) < \partial(BAA) = 3\partial(D) + 8\partial(C)$, $3\partial(D) > 2\partial(C)$ and $k \ge 21$, then $\partial(Z) < \partial(D) < \partial(C)$.

1. Suppose that $\overline{U_l} \equiv \overline{BAA} \equiv \overline{DCCDCCCDCCC}$ contains Z^k. Since Z^k doesn't occur in $\overline{CDCC}$, it doesn't occur in $\overline{DCCD}$ as well, since D is a beginning of C. Thus Z^k doesn't occur in any subword of $\overline{DCCDCCCDCCC}$ of length 4 over the alphabet $\{C, D\}$. Therefore, it covers either D^2 or C^2. Then, according to Lemma 1.2, either D or C is a proper power. It contradicts the inductive assumption.

2. Now let $\overline{V_l} \equiv \overline{BAAA} \equiv \overline{DCCDCCCDCCCDCCC}$ contain Z^k. In exactly the same way as for $\overline{U_l}$ we prove that either D or C is a proper power. The obtained contradiction proves Lemma 2.2.

Proof of Theorem 1. Let φ have finite order k in $\mathrm{Aut}(B(2,n))$, i.e. $\varphi^k \equiv id$. Then $a_1^{-1}X_k$ is equal to the empty word in $B(m,n)$. There are only two letters in the word $a_1^{-1}X_k$ that can be reduced. Therefore, according to Lemma 1.4, its cyclic irreducible form doesn't contain nonempty word Z^{18}. This contradicts the Lemma 5.5 of [5] and the point 1 of Theorem 2 of [6], according to which every word that is equal to empty word in $B(m,n)$ contains as a subword nonempty power Z^{1000}.

The proof of Theorem 2 repeats the proof of Theorem 1, changing «φ» by «ψ», «$a_1^{-1}X_k$» by «$a_1^{-1}U_k$ or $a_1^{-1}V_k$», «Z^{18}» by «Z^{21}» and referring to Lemma 2.2.

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Բավականաչափ մեծ պարբերությամբ ազատ պարբերական խմբերի անվերջ կարգի ավտոմորֆիզներ

Աշխատանքում կառուցվում են $m \geq 2$ ծնիչներով և բավականաչափ մեծ n պարբերությամբ $B(m,n)$ ազատ պարբերական խմբերի անվերջ կարգի ավտոմորֆիզմներ: Ստացված արդյունքներից հետևում է, որ $\mathrm{Aut}(B(m,n))$ խմբի քանորդ խումբն ըստ ներքին ավտոմորֆիզմների նորմալ ենթախմբի անվերջ է:

Автоморфизмы бесконечного порядка свободных периодических групп достаточно большого периода

В работе строятся автоморфизмы бесконечного порядка свободных периодических групп $B(m,n)$ достаточно большого периода n с $m \geq 2$ порождающими. Из полученных результатов следует, что фактор-группа группы $\mathrm{Aut}(B(m,n))$ по нормальной подгруппе внутренних автоморфизмов бесконечна.