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Mathematics

ON THE SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER IN COMPLEX DOMAIN

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The paper studies differential equations of fractional order of the form $D^{1/\rho}y(z)+\lambda y(z)=f(z)$ in the complex domain, where $\rho\ge 1$, λ is an arbitrary parameter, $D^{1/\rho}$ is the Riemann–Liouville differential operator. For functions of some classes Cauchy type problems are considered.

Keywords: Riemann–Liouville operators, differential equations of fractional order, complex domain.

§ 1. Preliminaries. Let $f(x)$ be an arbitrary function from the class $L(0,l)$, $0<l<+\infty$. The function $D^{-\alpha}f(x)=\dfrac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt$ is called the Riemann–Liouville integral of order α, $\alpha\in(0,+\infty)$, of function $f(x)$, and for $\alpha\in(0,1]$ the function $D^{\alpha}f(x)\equiv\dfrac{d}{dx}D^{-(1-\alpha)}f(x)$ is called the Riemann–Liouville derivative of order α of function $f(x)$.

It is known that in all Lebesgue points of $f(x)$, $\lim\limits_{\alpha\to+0}D^{-\alpha}f(x)=f(x)$ (and hence, almost everywhere) and, therefore, $\left[D^{-\alpha}f(x)\right]_{\alpha=0}=f(x)$ and $D^{1}f(x)=f'(x)$.

Let $\alpha\in[0,1)$, $\dfrac{1}{\rho}=1-\alpha$ $(\rho\ge 1)$, $x\in(0,l)$. The operators $D^{0}f(x)\equiv f(x)$, $D^{1/\rho}f(x)\equiv\dfrac{d}{dx}D^{-\alpha}f(x),\dots,D^{n/\rho}f(x)=D^{1/\rho}D^{(n-1)/\rho}f(x)$, $n\ge 2$, are called Riemann–Liouville operators of successive differentiation of order n/ρ of function $f(x)$. For more information on Riemann–Liouville operators see [1] (ch. IX) and [2] (§ 2).

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We introduce some notations. Following M.M. Djrbashian we denote $\Delta_\rho = \left\{ z; \left|\mathrm{Arg} z\right| < \frac{\pi}{2\rho}, 0 < |z| < \infty \right\}$, this domain is evidently many-sheeted for $0 < \rho < 1/2$ and is arranged on the Riemann supface G_∞ of the function Lnz. $H(\rho)$ is the class of functions $f(z)$ that are analytic in the domain Δ_ρ. Let us agree to denote $(0; l(\varphi)) = \{ z;\ \arg z = \varphi, 0 < |z| < l < +\infty \}$, $-\pi \le \varphi < \pi$.

Let $\alpha \in [0,1)$, $\frac{1}{\rho} = 1 - \alpha$ $(\rho \ge 1)$, $f(z)$ be an arbitrary function of a complex variable and $\left| f\left(re^{i\varphi}\right)\right| \in L\left(0; l(\varphi)\right), |\varphi| \le \pi, 0 < r < \infty.$ The function $D^{-\alpha} f(z) \equiv \frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} f(\xi) d\xi$, where integration is taken along the intercept connecting points 0 and z, $\arg(z-\xi)^{\alpha-1} = (\alpha-1)\arg z$, is called the Riemann–Liouville integral of order α of function $f(z)$, and the function $D^{1/\rho} f(z) \equiv \frac{d}{dz} D^{-\alpha} f(z)$ is called the Riemann–Liouville derivative of order $1/\rho$ of function $f(z)$.

The operators $D^{0/\rho} f(z) \equiv f(z)$, $D^{1/\rho} f(z) \equiv \frac{d}{dz} D^{-\alpha} f(z), \ldots, D^{n/\rho} f(z) = = D^{1/\rho} D^{(n-1)/\rho} f(z)$, $n \ge 2$, are called Riemann–Liouville operators of successive differentiation of order n/ρ of the function $f(z)$.

Function of Mittag–Leffler type is an entire function of the form $E_\rho(z;\mu) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu + n\rho^{-1})}$ $(\rho > 0)$ of order ρ with arbitrary value of parameter μ ([1], chap. VI, § 1).

For any $\mu > 0, \alpha > 0$ the following formula holds (see [1], ch. III (1.16)):

$$\frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} E_\rho(\lambda \xi^{1/\rho}; \mu) \xi^{\mu-1} d\xi = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho}; \mu+\alpha),\ z = re^{i\varphi}, \tag{1.1}$$

$$\xi = \tau e^{i\varphi},\ 0 < \tau < r < l < +\infty,\ -\pi \le \varphi < \pi,$$

λ is an arbitrary parameter.

§ 2. Main Results.

Theorem 2.1. Let $\alpha \in [0,1)$, $1/\rho = 1 - \alpha$ $(\rho \ge 1)$, λ is an arbitrary parameter. Then in the class of functions $\left| y(re^{i\varphi}) \right| \in L(0; l(\varphi))$, $\left| D^{-\alpha} y(re^{i\varphi}) \right| \in L(0; l(\varphi))$ the following problem of Cauchy type

$$D^{1/\rho} y(z) + \lambda y(z) = 0, \tag{2.1}$$

$$D^{-\alpha} y(z)|_{z=0} = 0 \tag{2.2}$$

has a unique solution

$$y(z) \equiv 0. \tag{2.3}$$

Proof. We note that for $z = x \in (0;+\infty)$ the similar problem is considered in the work [3] (see (3.4')– (3.5')). It is obvious that $y(z) \equiv 0$ is a solution of problem (2.1)–(2.2). We show that the solution (2.3) is unique. We note that $D^{1/\rho} y(z) \equiv \frac{d}{dz} D^{-\alpha} y(z) = -\lambda y(z)$, since $D^{-\alpha} y(z)|_{z=0} = 0$, then

$$D^{-\alpha} y(z) = -\lambda \int_0^z y(\xi) d\xi = -\lambda D^{-1} y(z), \quad \xi = \tau e^{i\varphi}, \; z = re^{i\varphi}. \tag{2.4}$$

Using the properties of fractional integrals and derivatives, from (2.4) we get that

$$y(z) = -\lambda D^{\alpha} D^{-1} y(z) = -\lambda D^{-\frac{1}{\rho}} y(z) \quad \text{(almost everywhere)}. \tag{2.5}$$

Using properties of fractional integrals from (2.5) we conclude that for any $P \geq 1$ the following identities also hold:

$$\begin{aligned} y(z) &= (-\lambda)^P D^{-\frac{P}{\rho}} y(z) = \frac{(-\lambda)^P}{\Gamma(P/\rho)} \int_0^z (z-\xi)^{\frac{P}{\rho}-1} y(\xi) d\xi = \\ &= \frac{(-\lambda)^P}{\Gamma(P/\rho)} \int_0^r \left(re^{i\varphi} - \tau e^{i\varphi}\right)^{\frac{P}{\rho}-1} y\left(\tau e^{i\varphi}\right) e^{i\varphi} d\tau = \frac{(-\lambda)^P}{\Gamma(P/\rho)} e^{\frac{i\varphi P}{\rho}} \int_0^r (r-\tau)^{\frac{P}{\rho}-1} y\left(\tau e^{i\varphi}\right) d\tau. \end{aligned} \tag{2.6}$$

From (2.6) for $P \geq \rho$ we get

$$\max_{0 \leq r \leq l} \left| y\left(re^{i\varphi}\right) \right| \leq \frac{l^{-1}\left(l^{1/\rho}|\lambda|\right)^P}{\Gamma(P/\rho)} \int_0^l \left| y\left(\tau e^{i\varphi}\right) \right| d\tau. \tag{2.7}$$

But since $\left| y\left(re^{i\varphi}\right) \right| \in L(0; l(\varphi))$ as $P \to \infty$, then from (2.7) follows the statement of Theorem, i.e. $y(z) \equiv 0$.

Theorem 2.2. Let $\alpha \in [0,1)$, $\frac{1}{\rho} = 1 - \alpha \quad (\rho \geq 1)$, λ is an arbitrary parameter. Then the function

$$y(z;\lambda) = e_\rho(z;\lambda) \equiv E_\rho\left(-\lambda z^{1/\rho}; 1/\rho\right) z^{\frac{1}{\rho}-1}, \tag{2.8}$$

is the solution of the following Cauchy type problem:

$$D^{1/\rho} y(z) + \lambda y(z) = 0, \tag{2.9}$$

$$D^{-\alpha} y(z)|_{z=0} = 1. \tag{2.10}$$

Proof. We note that for $z = x \in (0;+\infty)$ the Theorem 2.2 is true (see [2, 3]). By the definition of operator $D^{1/\rho}$ we have $D^{1/\rho} y(z;\lambda) \equiv \frac{d}{dz} D^{-\alpha} y(z;\lambda)$, where

$$\begin{aligned} D^{-\alpha} y(z;\lambda) &\equiv \frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} y(\xi;\lambda) d\xi = \\ &= \frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} E_\rho\left(-\lambda \xi^{1/\rho}; 1/\rho\right) \xi^{\frac{1}{\rho}-1} d\xi \left(z = re^{i\varphi}, \, \xi = \tau e^{i\varphi}\right). \end{aligned} \tag{2.11}$$

Using the formula (1.1), from (2.11) we get

$$D^{-\alpha}y(z;\lambda)=E_\rho\left(-\lambda z^{1/\rho};1\right). \tag{2.12}$$

But it can be easily proved that $\frac{d}{dz}\left\{E_\rho\left(-\lambda z^{1/\rho};1\right)\right\}=-\lambda E_\rho\left(-\lambda z^{1/\rho};1/\rho\right)z^{\frac{1}{\rho}-1}$, and hence

$$D^{1/\rho}y(z;\lambda)\equiv\frac{d}{dz}D^{-\alpha}y(z;\lambda)=-\lambda E_\rho\left(-\lambda z^{1/\rho};1/\rho\right)z^{\frac{1}{\rho}-1}=-\lambda y(z;\lambda),$$

i.e. $D^{1/\rho}y(z;\lambda)+\lambda y(z;\lambda)\equiv 0$ and $D^{-\alpha}y(z;\lambda)|_{z=0}=1$.

Now we show that the solution (2.8) is unique. We suppose that the problem (2.9)–(2.10) has another solution $\tilde{y}(z;\lambda)$. We put $y^*(z;\lambda)\equiv y(z;\lambda)-\tilde{y}(z;\lambda)$. Then it can be easily seen that the function $y^*(z;\lambda)$ is a solution of the problem (2.1)–(2.2), consequently $y^*(z;\lambda)\equiv 0$, i.e. $y(z;\lambda)\equiv\tilde{y}(z;\lambda)$.

Theorem 2.2 is proved.

Theorem 2.3. Let $\alpha\in[0,1)$, $1/\rho=1-\alpha$ $(\rho\ge 1)$, λ is an arbitrary parameter, $f(z)\in H(\rho)$, $f(re^{i\varphi})\in L(0;l(\varphi))$.

Then the function

$$y(z;\lambda)=\int_0^z e_\rho(z-\xi;\lambda)f(\xi)d\xi,\ z=re^{i\varphi},\ \xi=\tau e^{i\varphi}, \tag{2.13}$$

where $e_\rho(z;\lambda)\equiv E_\rho\left(-\lambda z^{1/\rho};1/\rho\right)z^{\frac{1}{\rho}-1}$ is the solution of the following Cauchy type problem:

$$D^{1/\rho}y(z)+\lambda y(z)=f(z), \tag{2.14}$$

$$D^{-\alpha}y(z)|_{z=0}=0. \tag{2.15}$$

Proof. Let $z=re^{i\varphi}$, $\xi=\tau e^{i\varphi}$, $\xi_1=\tau_1 e^{i\varphi}$, $0<\tau_1<\tau<r<l<+\infty$. By the definition of operator $D^{-\alpha}$ we have:

$$\begin{aligned}D^{-\alpha}y(z;\lambda)&\equiv\frac{1}{\Gamma(\alpha)}\int_0^z(z-\xi)^{\alpha-1}d\xi\int_0^\xi e_\rho(\xi-\xi_1;\lambda)f(\xi_1)d\xi_1=\\&=\frac{1}{\Gamma(\alpha)}\int_0^r\left(re^{i\varphi}-\tau e^{i\varphi}\right)^{\alpha-1}e^{i\varphi}d\tau\int_0^\tau e_\rho\left(e^{i\varphi}(\tau-\tau_1);\lambda\right)f\left(\tau_1e^{i\varphi}\right)e^{i\varphi}d\tau_1=\\&=\frac{e^{i\varphi(\alpha+1)}}{\Gamma(\alpha)}\int_0^r(r-\tau)^{\alpha-1}d\tau\int_0^\tau e_\rho\left(e^{i\varphi}(\tau-\tau_1);\lambda\right)f\left(\tau_1e^{i\varphi}\right)d\tau_1=\\&=\frac{e^{i\varphi(\alpha+1)}}{\Gamma(\alpha)}e^{i\left(\frac{1}{\rho}-1\right)\varphi}\int_0^r(r-\tau)^{\alpha-1}d\tau\int_0^\tau e_\rho(\tau-\tau_1;\lambda^*)f\left(\tau_1e^{i\varphi}\right)d\tau_1=\\&=\frac{e^{i\varphi}}{\Gamma(\alpha)}\int_0^r(r-\tau)^{\alpha-1}d\tau\int_0^\tau e_\rho(\tau-\tau_1;\lambda^*)f\left(\tau_1e^{i\varphi}\right)d\tau_1,\end{aligned} \tag{2.16}$$

where $\lambda^*=\lambda e^{i(\varphi/\rho)}$, $e_\rho\left(e^{i\varphi}(\tau-\tau_1);\lambda\right)=e^{i\left(\frac{1}{\rho}-1\right)\varphi}e_\rho(\tau-\tau_1;\lambda^*)$.

We put $f\left(\tau_1 e^{i\varphi}\right)=U_{\varphi}\left(\tau_1\right)+iV_{\varphi}\left(\tau_1\right)$,

$$I_1 \equiv \frac{e^{i\varphi}}{\Gamma(\alpha)}\int_0^r (r-\tau)^{\alpha-1}d\tau\int_0^{\tau} e_{\rho}\left(\tau-\tau_1;\lambda^*\right)U_{\varphi}\left(\tau_1\right)d\tau_1, \tag{2.17}$$

$$I_2 \equiv \frac{ie^{i\varphi}}{\Gamma(\alpha)}\int_0^r (r-\tau)^{\alpha-1}d\tau\int_0^{\tau} e_{\rho}\left(\tau-\tau_1;\lambda^*\right)V_{\varphi}\left(\tau_1\right)d\tau_1. \tag{2.18}$$

In the formulas (2.17), (2.18), by changing the order of integration and using the formula (1.1), we get

$$I_1 \equiv \frac{e^{i\varphi}}{\Gamma(\alpha)}\int_0^r (r-\tau)^{\alpha-1}d\tau\int_0^{\tau} e_{\rho}\left(\tau-\tau_1;\lambda^*\right)U_{\varphi}\left(\tau_1\right)d\tau_1=\frac{e^{i\varphi}}{\Gamma(\alpha)}\int_0^r U_{\varphi}\left(\tau_1\right)d\tau_1\int_{\tau_1}^{\tau}(r-\tau)^{\alpha-1}e_{\rho}\left(\tau-\tau_1;\lambda^*\right)d\tau=$$

$$=\frac{e^{i\varphi}}{\Gamma(\alpha)}\int_0^r U_{\varphi}\left(\tau_1\right)d\tau_1\int_0^{r-\tau_1}\left(\left(r-\tau_1\right)-v\right)^{\alpha-1}e_{\rho}\left(v;\lambda^*\right)dv=e^{i\varphi}\int_0^r U_{\varphi}\left(\tau_1\right)E_{\rho}\left(-\lambda^*\left(r-\tau_1\right)^{1/\rho};1\right)d\tau_1 \tag{2.19}$$

and

$$I_2 = ie^{i\varphi}\int_0^r V_{\varphi}\left(\tau_1\right)E_{\rho}\left(-\lambda^*\left(r-\tau_1\right)^{1/\rho};1\right)d\tau_1. \tag{2.20}$$

Moreover, according to (2.19), (2.20), from (2.16) we obtain

$$D^{-\alpha}y(z;\lambda)\equiv I_1+I_2=e^{i\varphi}\int_0^r E_{\rho}\left(-\lambda^*\left(r-\tau_1\right)^{1/\rho};1\right)f\left(\tau_1 e^{i\varphi}\right)d\tau_1. \tag{2.21}$$

Now using the formula (2.21), we will have

$$\frac{d}{dz}D^{-\alpha}y(z;\lambda)=\frac{d}{dr}\left(e^{i\varphi}\int_0^r E_{\rho}\left(-\lambda^*\left(r-\tau_1\right)^{1/\rho};1\right)f\left(\tau_1 e^{i\varphi}\right)d\tau_1\right)\frac{dr}{dz}=$$

$$=f\left(re^{i\varphi}\right)-\lambda^*\int_0^r E_{\rho}\left(-\lambda^*\left(r-\tau_1\right)^{1/\rho};1/\rho\right)\left(r-\tau_1\right)^{\frac{1}{\rho}-1}f\left(\tau_1 e^{i\varphi}\right)d\tau_1=$$

$$=f(z)-\lambda e^{i(\varphi/\rho)}\int_0^r E_{\rho}\left(-\lambda e^{i(\varphi/\rho)}\left(r-\tau_1\right)^{1/\rho};1/\rho\right)\left(r-\tau_1\right)^{\frac{1}{\rho}-1}f\left(\tau_1 e^{i\varphi}\right)d\left(\tau_1 e^{i\varphi}\right)=$$

$$=f(z)-\lambda\int_0^r E_{\rho}\left(-\lambda\left(re^{i\varphi}-\tau_1 e^{i\varphi}\right)^{1/\rho};1/\rho\right)\left(re^{i\varphi}-\tau_1 e^{i\varphi}\right)^{\frac{1}{\rho}-1}f\left(\tau_1 e^{i\varphi}\right)d\left(\tau_1 e^{i\varphi}\right)=$$

$$=f(z)-\lambda\int_0^z E_{\rho}\left(-\lambda\left(z-\xi_1\right)^{1/\rho};1/\rho\right)\left(z-\xi_1\right)^{\frac{1}{\rho}-1}f\left(\xi_1\right)d\xi_1=$$

$$=f(z)-\lambda\int_0^z e_{\rho}\left(z-\xi_1;\lambda\right)f\left(\xi_1\right)dt_1=f(z)-\lambda y(z;\lambda),$$

i.e. $D^{1/\rho}y(z;\lambda)+\lambda y(z;\lambda)\equiv f(z)$ and $D^{-\alpha}y(z;\lambda)|_{z=0}=0$.

Now we show, that the solution is unique. We suppose that the problem (2.14)–(2.15) has another solution $\tilde{y}(z;\lambda)$. We put $y^*(z;\lambda)\equiv y(z;\lambda)-\tilde{y}(z;\lambda)$.

It can be easily proved that $y^*(z;\lambda)$ is the solution of the problem (2.1)–(2.2). But, according to Theorem 2.1, we have $y^*(z;\lambda)\equiv 0$, i.e. $y(z;\lambda)\equiv\tilde{y}(z;\lambda)$.

Theorem 2.3 is proved.

Theorem 2.4. Let $\alpha \in [0,1)$, $\frac{1}{\rho} = 1-\alpha$ $(\rho \geq 1)$, λ is an arbitrary parameter, $f(z) \in H(\rho)$, $f(re^{i\varphi}) \in L(0; l(\varphi))$.

Then the function

$$y(z;\lambda) = a_0 e_\rho(z;\lambda) + \int_0^z e_\rho(z-\xi;\lambda) f(\xi) d\xi, \quad z = re^{i\varphi}, \ \xi = \tau e^{i\varphi}, \tag{2.22}$$

is the solution of the following Cauchy type problem:

$$D^{1/\rho} y(z) + \lambda y(z) = f(z), \tag{2.23}$$

$$D^{-\alpha} y(z)|_{z=0} = a_0. \tag{2.24}$$

Proof. We note that from of the Theorems (2.2)–(2.3) we have $(D^{1/\rho} + \lambda) e_\rho(z;\lambda) \equiv 0$, and $(D^{1/\rho} + \lambda)\left(\int_0^z e_\rho(z-\xi;\lambda) f(\xi) d\xi\right) \equiv f(z)$.

It is easy to see, that $D^{-\alpha} y(z;\lambda)|_{z=0} = a_0$.

Consequently, the function (2.22) is a solution of the problem (2.23)–(2.24). Now we show that the solution is unique.

Let the function $\tilde{y}(z;\lambda)$ is a solution of the problem (2.23)–(2.24).

We put $y^*(z;\lambda) \equiv y(z;\lambda) - \tilde{y}(z;\lambda)$. It is easy to see that the function $y^*(z;\lambda)$ is the solution of the problem (2.1)–(2.2). Consequently, $y^*(z;\lambda) \equiv 0$, i.e. $y(z;\lambda) \equiv \tilde{y}(z;\lambda)$.

Theorem (2.4) is proved.

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Բ. Հ. Սահակյան

Կոմպլեքս տիրույթում կոտորակային կարգի որոշ դիֆերենցիալ հավասարումների լուծումների մասին

Այս աշխատանքում դիտարկվում են կոմպլեքս տիրույթում $D^{1/\rho}y(z)+\lambda y(z)=f(z)$, տեսքի կոտորակային կարգի դիֆերենցիալ հավասարումներ, որտեղ $\rho\geq 1$, λ -ն կամայական պարամետր է, $D^{1/\rho}$ -ն Ռիման–Լիուվիլի դիֆերենցիալ օպերատորն է: Որոշ դասի ֆունկցիաների համար դիտարկվում և լուծվում են Կոշիի տիպի խնդիրներ:

Б. А. Саакян.

О решениях некоторых дифференциальных уравнений дробного порядка в комплексной области

В работе в комплексной области рассматриваются уравнения дробного порядка вида $D^{1/\rho}y(z)+\lambda y(z)=f(z)$, где $\rho\geq 1$, λ – произвольный параметр, $D^{1/\rho}$ – дифференциальный оператор Римана–Лиувилля. Для функций некоторых классов рассматриваются и решаются задачи типа Коши.