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Mathematics

A REMARK ON STRICT UNIFORM ALGEBRAS

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We study some properties of algebras of bounded continuous functions on a completely regular space, these algebras being equipped with the strong topology defined by of family multiplication operators (strict uniform algebras). We prove an analog of a theorem due to M. Sheinberg for strict uniform algebras (see [1–3]).

***Keywords*:** Strict uniform algebra, amenable algebra, bimodule.

Let Ω be a completely regular Hausdorff space, and $C_*(\Omega)$ be the algebra of all bounded complex-valued continuous functions on Ω. If we equip the space $C_*(\Omega)$ with the topology induced by sup-norm $\|f\|_\infty = \sup\{|f(x)|: x \in \Omega\}$, then we obtain a commutative Banach algebra $C_b(\Omega)$ with the property that the maximal ideals space of which is $M_{C_b(\Omega)} = \beta\Omega$, where $\beta\Omega$ is the Stone-Chekh compactification for Ω. Recall that we call *the remainder* of Ω in the extension $\beta\Omega$ the space $\beta\Omega \setminus \Omega$ with the topology induced from $\beta\Omega$ (see [4–5]). Let $\mathcal{K}(\Omega)$ be the set of all compacts $Q \subset \beta\Omega \setminus \Omega$ and for $Q \in \mathcal{K}(\Omega)$ denote

$$C_Q = C_Q(\Omega) = \{f \in C_b(\Omega) : \hat{f}|_Q = 0\},$$

where $\hat{f}$ is the Gelfand transform of f. Then $C_Q(\Omega)$ is Banach algebra with bounded approximative identity, and $C_b(\Omega)$ is C_Q-module. In the case when $Q_1, Q_2 \in \mathcal{K}(\Omega)$ and $Q_1 \subset Q_2$, we have $C_{Q_1}(\Omega) \supset C_{Q_2}(\Omega)$.

Note that the remainder $\beta\Omega \setminus \Omega$ has a rather complicated structure, because, for instance, in every point of the remainder the first axiom of countability fails to hold. For $Q \in \mathcal{K}(\Omega)$ denote $\Omega_Q = \beta\Omega \setminus Q$. All the Banach algebras $C_Q(\Omega)$ are proper closed ideals in the algebra $C_b(\Omega)$ for every $Q \in \mathcal{K}(\Omega)$.

Every ideal $C_Q(\Omega)$ defines a family of seminorms $\{P_g\}_{g \in C_Q(\Omega)}$ on $C_b(\Omega)$, with $P_g(f) = \|T_g f\|_\infty$, where $T_g : C_b(\Omega) \to C_b(\Omega)$ is the multiplicative operator

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$T_g f = gf$. The topology on $C_b(\Omega)$, defined by this family of seminorms, we will call the β_Q*-topology*, and we will denote by $C(\Omega)_{\beta_Q}$ the algebra $C_b(\Omega)$ endowed with the β_Q-topology (cf. [6–9]). It is easy to see that β_Q-topology is Hausdorff topology.

We will say that a closed in the β_Q-topology subalgebra $\mathcal{A}$ of algebra $C(\Omega)_{\beta_Q}$ is β_Q*-uniform*, if it contains constants and separates the points of Ω_Q (i.e. for any $x_1, x_2 \in \Omega_Q$ with $x_1 \neq x_2$, there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$).

Note that in the case of completely regular space Ω, the ideal $C_0(\Omega)$ can turn out to be unusable, because of its triviality.

It should be noted here that β-topology of Buck on $C_*(\Omega)$ is the inductive limit $Lind(\beta_Q)$ of β_Q-topologies for $Q \in \mathcal{K}(\Omega)$.

If $Q \in \mathcal{K}(\Omega)$, then $\Omega_Q = \beta\Omega \setminus Q$ is locally-compact Hausdorff space and in that case one can introduce a strong topology on $C_*(\Omega_Q)$ using the ideal $C_0(\Omega_Q)$, which we will denote by $C(\Omega_Q)_\beta$.

Since $\Omega \subset \Omega_Q \subset \beta\Omega$, then the space of maximal ideals $M_{C_b(\Omega_Q)} = \beta(\Omega_Q) = \beta(\Omega)$.

It can be easily seen, that the algebra $C(\Omega)_{\beta_Q}$ is topologically isomorphic to the algebra $C(\Omega_Q)_\beta$ and hence the following assertions hold (cf. [2, 3]):

Theorem 1.

a) For any $Q \in \mathcal{K}(\Omega)$ the algebra $C(\Omega)_{\beta_Q}$ is β_Q-complete locally convex algebra;

b) $C_0(\Omega_Q) = C_Q(\Omega)$ is everywhere dense in $C(\Omega)_{\beta_Q}$;

c) the space of all β_Q-continuous linear functionals on $C(\Omega)_{\beta_Q}$ is isomorphic to the space $M(\Omega_Q)$ of all finite regular measures on Ω_Q.

Proposition 1.

a) The uniform topology and β_Q-topology on $C_0(U) = \{f \in C_b(\Omega) : f|_{\Omega\setminus U} = 0\}$ coincide for every open set U in Ω_Q such that $\overline{U} \subset \Omega_Q$.

b) The linear space generated by $\{C_0(U_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is the subset of the set of all open subsets in Ω_Q such that $U_i \subset \overline{U_i} \subset \Omega_Q$, is β_Q-dense in $C(\Omega_Q)_\beta \simeq C(\Omega)_{\beta_Q}$.

Let A be a β_Q-uniform algebra on Ω. Since the algebra $C_b(\Omega_Q)$ is complete in the β_Q-topology, then A is a closed subalgebra of the algebra $C_b(\Omega_Q)$ in the sup-norm. Hence, we will denote the algebra A in the sup-norm of $C_b(\Omega_Q)$ by $A_{b,Q}$.

Suppose that the Banach space X is $A_{b,Q}$-bimodule. Recall that X is β_Q-complete $A_{b,Q}$-bimodule, if from the fact that the net $\{f_i\}_{i\in I}$ in A β_Q-converges to f_0 it follows that for any $x\in X$ the nets $\{f_i x\}_{i\in I}$ and $\{xf_i\}_{i\in I}$ converge to $f_0 x$ and xf_0 respectively in the norm of the Banach space X.

The bimodular operation on X defines a bimodular operation on the dual space X^* of X

$$(f\varphi)(x)=\varphi(xf),\quad (\varphi f)(x)=\varphi(fx)$$

for all $f\in A$, $x\in X$, $\varphi\in X^*$.

Note also that linear functional $\varphi\in X^*$ is called *weak** β_Q*-continuous*, if from the β_Q-convergence in A of the net $\{f_i\}_{i\in I}$ to f_0 it follows that the net of functionals $\{f_i\varphi\}_{i\in I}$ and $\{\varphi f_i\}_{i\in I}$ converge in the weak topology to $f_0\varphi$ and φf_0 respectively.

As in ([2, 3]) we define the abelian group $Z^1_{\beta_Q}(A,X^*)$ of all β_Q-continuous in the weak* topology differentiations $D:A\to X^*$ (i.e. if the net $\{f_i\}_{i\in I}$ in A β_Q-converges to f_0, then the net of functionals $\{D(f_i)\}_{i\in I}$ converges to $D(f_0)$ in the weak* topology of X^*). We denote by $Z^1_*(A,X^*)$ the abelian group of all continuous in the weak* topology differentiations $D:A_{b,Q}\to X^*$. For every $Q\in\mathcal{K}(\Omega)$, $Z^1_{\beta_Q}(A,X^*)$ is a subgroup of $Z^1(A,X^*)$.

Following B. Johnson [10] one calls a Banach algebra $A_{b,Q}$ to be *amenable*, if the group $H^1(A,X^*)=Z^1(A,X^*)/B^1(A,X^*)$ is trivial for every $A_{b,Q}$-bimodule X, where $B^1(A,X^*)$ is the abelian group, consisting of all inner differentiations $\delta_\varphi(a)=a\varphi-\varphi a$. Analogously, the algebra A is called β_Q*-amenable*, if the group $H^1_{\beta_Q}(A,X^*)=Z^1_{\beta_Q}(A,X^*)/B^1_{\beta_Q}(A,X^*)$ is trivial for any β_Q-complete $A_{b,Q}$-bimodule X.

Clearly, if A is an amenable algebra, then A is β_Q-amenable (i.e. from the condition $H^1(A,X^*)=0$ for any $A_{b,Q}$-bimodule X it follows that $H^1_{\beta_Q}(A,X^*)=0$ for any β_Q-complete $A_{b,Q}$-bimodule X).

For the rest we need two β_Q-complete $A_{b,Q}$-bimodules.

Proposition 2. Let $\mu\in M(\Omega_Q)$. Then there exists a measure $\nu\in M(\Omega_Q)$ and a function $g\in C_0(\Omega_Q)$ such that $\mu=g^*\nu$, i.e. $\int fd\mu=\int fgd\nu$ for all $f\in C_0(\Omega_Q)$ $(\simeq C_Q(\Omega))$.

Theorem 2. For any positive measure $\mu\in M(\Omega_Q)$ the Hilbert space $L^2(\Omega_Q,\mu)$ is β_Q-complete Banach $A_{b,Q}$-bimodule.

The proof can be done in the same manner as of the Lemma 4 in [3].

Let $B_Q = BL(L^2(\Omega_Q, \mu))$ be the algebra of all bounded linear operators in $L^2(\Omega_Q, \mu)$, and $J_{1,Q}$ be the ideal of nuclear operators, which is Banach space in the nuclear norm $\|T\|_1 = \mathrm{tr}|T|$ ([11]). $J_{1,Q}$ becomes Banach $A_{b,Q}$-bimodule in the case $f \cdot T \cdot g = T_f \cdot T \cdot T_g$ for all $f, g \in A_{b,Q}$ and $T \in J_{1,Q}$.

It is easy to see (c.f. [11]), that for any $T \in J_{1,Q}$ there exists a positive function $g \in C_0(\Omega_Q)$ such that $T_{g^{-1}} \cdot T \in J_{1,Q}$.

Theorem 3. The Banach space $J_{1,Q}$ is β_Q-complete $A_{b,Q}$-bimodule.

It is well known, that the algebra B_Q is isometrically isomorphic, as a Banach space, to the dual space $J^*_{1,Q}$ (c.f. [9]). This leads to the following result.

Theorem 4. The Banach $A_{b,Q}$-bimodule B_Q is isometrically isomorphic as a $A_{b,Q}$-bimodule to the β_Q-complete in the weak* topology Banach $A_{b,Q}$-bimodule $J^*_{1,Q}$.

Using Lemma 7 form [3], one can analogously prove the following

Proposition 3. Let A be β_Q-complete uniform algebra. If $A \neq C(\Omega)_{\beta_Q}$, then $H^1_{\beta_Q}(A, X^*) \neq 0$ for some β_Q-complete Banach $A_{b,Q}$-bimodule X.

From this Proposition we get the following result, which is the main result of the paper.

Theorem 5. Let A be β_Q-uniform algebra. Then the following conditions are equivalent:

a) $A = C(\Omega)_{\beta_Q}$;

b) A is amenable algebra;

c) A is β_Q-amenable algebra.

Now consider the situation, when Ω is completely regular Hausdorff space. In this case, as has been mentioned above, one can introduce β-topology in the algebra $C_*(\Omega)$ as the inductive limit $\mathrm{Lind}_Q(\beta_Q)$ of β_Q-topologies, where $Q \in \mathcal{K}(\Omega)$, which we will denote again by $C(\Omega)_\beta$. Then by β-uniform algebra A over Ω we will mean (as above) a closed in the β-topology subalgebra in the algebra $C(\Omega)_\beta$, which contains constants and separates the points of Ω.

It is easy to see, that β-topology on A is the inductive limit $\mathrm{Lind}\left(\beta_Q\right)$ of β_Q-topologies of algebras A_{β_Q}, which are β_Q-uniform subalgebras of algebras $C(\Omega)_{\beta_Q}$ respectively.

In the light of the obtained results, we can formulate the following results for completely regular Hausdorff space Ω.

Theorem 6.
a) The algebra $C(\Omega)_\beta$ is β-complete locally convex algebra;

b) the space of all β-continuous linear functionals on $C(\Omega)_\beta$ is isomorphic to the space $M(\Omega)$ of all finite regular measures on Ω.

Theorem 7. Let A be β-uniform subalgebra of $C(\Omega)_\beta$. Then the following conditions are equivalent:

a) $A = C(\Omega)_\beta$;

b) A is amenable algebra.

In the case, when Ω is a compact, we get the Theorem of M. Sheinberg from [1].

Remark. Note that *null-set* is a set of the form $f^{-1}(0)$ with $f \in C_*(\Omega)$. Let $\mathcal{Z}(\Omega)$ is the set of all null-sets $Z \in \beta\Omega \setminus \Omega$. If $Z \in \mathcal{Z}(\Omega)$, then $\beta\Omega \setminus Z$ is σ-compact and locally-compact space and, in the light of Theorem 2.6 from [12], in $C_*(\beta\Omega \setminus Z)$ the strong topology coincides with the strong topology of Mackey (i.e. strong space of Mackey). It follows that $C(\beta\Omega \setminus Z)_\beta$ is $C(\Omega)_{\beta_Z}$. Hence all the above idealogy works also for β_Z-uniform algebras.

Note that in the algebra $C_*(\Omega)$ one can introduce also the β_1-topology as the inductive limit $\mathrm{Lind}_Z(\beta_Z)$ of β_Z-topologies, where $Z \in \mathcal{Z}(\Omega)$, which we will denote by $C(\Omega)_{\beta_1}$. This β_1-topology, as well as β-topology, is locally convex, Hausdorff and $\beta \le \beta_1 \le \|\cdot\|$. For β_1-uniform algebras over Ω the analogues of Theorem 6 and Theorem 7 are true.

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REFERENCES

1. **Sheinberg M.V.** Uspekhi Mat. Nauk, 1977, v. 32, № 5 (197), p. 203–204 (in Russian).
2. **Karakhanyan M.I., Khor'kova T.A.** Funktsional. Anal. i Prilozhen., 2009, v. 43, № 1, p. 85–87. Translation in Funct. Anal. Appl., 2009, v. 43, № 1, p. 69–71 (in Russian).
3. **Karakhanyan M.I., Khor'kova T.A.** Sibirsk. Mat. Zh., 2009, v. 50, № 1, p. 96–106. Translation in Sib. Math. J., 2009, v. 50, № 1, p. 77–85 (in Russian).
4. **Gelfand I.M., Raikov D.A., Shilov G.E.** Commutative Normed Rings. M.: Izd. Fiz.-Mat. Lit., 1960 (in Russian).
5. **Arkhangelskii A.V., Ponomarev V.I.** Fundamentals of General Topology in Problems and Exercise. M.: Nauka, 1974 (in Russian).
6. **Buck R.C.** Michigan Math. J., 1958, v. 5, № 2, p. 95–104.
7. **Sentilles F.D.** Trans. Amer. Math. Soc., 1972, v. 168, p. 311–336.
8. **Giles R.** Trans. Amer. Math. Soc., 1971, v. 161, p. 467–474.
9. **Glicksberg I.** Proc. Amer. Math. Soc., 1963, v. 14, p. 329–333.
10. **Johnson B.E.** Memoirs of the American Mathematical Society, № 127. American Mathematical Society. Providence, R.I., 1972, 96 p.
11. **Reed M., Simon B.** Methods of modern mathematical physics. V. 1. New York, London: Functional Analysis Acad. Press, 1972.
12. **Conway J.B.** Trans. Amer. Math. Soc., 1967, v. 126, p. 474–486.

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Դիտողություն խիստ հավասարաչափ հանրահաշիվների վերաբերյալ

Աշխատանքում ուսումնասիրվում են որոշ հատկություններ լիովին ռեգուլյար (կանոնավոր) տարածության վրա որոշված սահմանափակ անընդհատ ֆունկցիաների մի հանրահաշվի, որում մտցված է բազմապատկման օպերատորների ընտանիքով առաջացած խիստ հավասարաչափ տոպոլոգիա: Ապացուցվում է Մ. Շեյնբերգի թեորեմը խիստ հավասարաչափ հանրահաշիվների համար:

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Замечания о строго равномерных алгебрах

В статье изучаются некоторые свойства алгебры ограниченных непрерывных функций на вполне регулярном пространстве, в которой введена строгая равномерная топология, порожденная семейством операторов умножения. Доказан аналог теоремы М. Шейнберга для строго равномерных алгебр.