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Mathematics

CLASSES OF TAYLOR–MACLOURIN TYPE FORMULAE IN COMPLEX DOMAIN

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In the present paper some systems of operators generated by Riemann–Liouville integral and derivative of orders $\alpha \in (0,1)$, and functions generated by Mittag–Leffler type functions are introduced.

Some properties of these systems are investigated and for a certain class of functions the generalizations of Taylor–Maclourin type formulae are obtained.

***Keywords*:** Riemann–Liouville operators, Taylor–Maclourins type formulae.

§ 1. Introduction. In [1] the authors defined a system of functions

$$\varepsilon_n^{(\rho)}(x) \equiv \frac{\rho}{2\pi i} \int_{\gamma(\varepsilon;\beta)} \frac{e^{x\xi^\rho} \xi^{\rho-1}}{\prod_{j=0}^{n} (\xi + \lambda_i)} d\xi, \quad n \ge 0, \; x \in (0,+\infty),$$

as well as a systems of operators associated with the Riemann–Liouville integrodifferential operators D^γ: $L^{0/\rho}(f) \equiv f(x)$, $L^{n/\rho} f(x) \equiv \prod_{j=0}^{n-1} (D^{1/\rho} + \lambda_j) f(x)$, $\tilde{L}^{n/\rho} f(x) \equiv D^{-\alpha} L^{n/\rho} f(x)$, $n \ge 1$, where $\rho \ge 1$, $1-\alpha = 1/\rho$, $\lambda_{j+1} \ge \lambda_j \ge 0$, $j = 0,1,\dots$, $\gamma(\varepsilon;\beta)$ is a contour in plane ξ ([2], p. 126).

The functions $\varepsilon_n^{(\rho)}(x)$ admit representations of the form (see, [1])

$$\varepsilon_n^{(\rho)}(x) = \sum_{j=0}^{n} C_j^{(n)} E_\rho^{(S_j-1)}(-\lambda_j x^{1/\rho}; 1/\rho) x^{\frac{1}{\rho}-1}, \; n \ge 0,$$

where $S_j (S_j \ge 1)$ is the multiplicity, with which the number λ_j appears on the interval $\{\lambda_k\}_0^j$ of our sequence. Coefficients $\{C_k^{(n)}\}_0^n$ are determined from an expansion $R_n(\xi) = \left\{ \prod_{j=0}^{n} (\xi + \lambda_j) \right\}^{-1} = \sum_{j=0}^{n} \frac{\Gamma(S_j) C_j^{(n)}}{(\xi + \lambda_j) S_j}$, and $E_\rho(z;\mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}$, $\rho > 0$, is the ρ order entire function of Mittag–Leffler type for any value of parameter μ ([2], chap. VI, § 1).

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It is also known [1] that the functions $\varepsilon_n^{(\rho)}(x)$, $n \ge 0$, admit representation of the form

$$\varepsilon_n^{(\rho)}(x) \equiv \Omega_n^{(\rho)}(x) = \int_0^x e_\rho(x-t_1;\lambda_0)dt_1 \int_0^{t_1} e_\rho(t_1-t_2;\lambda_1)dt_2 \ldots \int_0^{t_{n-1}} e_\rho(t_{n-1}-t_n;\lambda_{n-1}) e_\rho(t_n;\lambda_n)dt_n,$$

where $e_\rho(x;\lambda) \equiv E_\rho(-\lambda x^{1/\rho};1/\rho)x^{\frac{1}{\rho}-1}$, $\rho \ge 1$, $x \in (0,+\infty)$.

Particularly, in [1] the following Taylor–Maclourins type formula was found for some classes of functions:

$$f(x) = \sum_{k=0}^n \tilde{L}^{k/\rho} f(0)\varepsilon_k^{(\rho)}(x) + \int_0^x \varepsilon_n^{(\rho)}(x-t)L^{\frac{n+1}{\rho}} f(t)dt, \; x \in (0,l).$$

In the present work similar problems are investigated in the complex domain.

§ 2. Preliminaries Information. Let $f(x) \in L(0,l)$ $(0 < l < +\infty)$, $\alpha \in (0,+\infty)$. The function $D^{-\alpha} f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$ is called the Riemann–Liouville integral of order α of function $f(x)$, and for $\alpha \in (0,1]$ the function $D^\alpha f(x) \equiv \frac{d}{dx} D^{-(1-\alpha)} f(x)$ is called the Riemann–Liouville derivative of order α of function $f(x)$. It is known that $\lim_{\alpha \to +0} D^{-\alpha} f(x) = f(x)$ (almost everywhere) in all Lebeg points $x \in (0,l)$ of the function $f(x)$ and, therefore, $[D^{-\alpha} f(x)]_{\alpha=0} = f(x)$ and $D' f(x) = f'(x)$.

Let $\alpha \in [0,1)$, $\frac{1}{\rho} = 1-\alpha$, $\rho \ge 1$, $x \in (0,l)$. The operators $D^0 f(x) \equiv f(x)$, $D^{1/\rho} f(x) \equiv \frac{d}{dx} D^{-\alpha} f(x)$, $D^{n/\rho} f(x) \equiv D^{\frac{1}{\rho}} D^{\frac{n-1}{\rho}} f(x)$, $n \ge 2$, are called Riemann–Liouville operators of successive differentiation of order n/ρ of function $f(x)$ [2, 3].

Introduce some notations. We denote $\Delta_\rho = \left\{ z; |\operatorname{Arg} z| < \frac{\pi}{2\rho}, 0 < |z| < +\infty \right\}$. This domain for $0 < \rho < 1/2$ being evident by a manifold and arranged on the Riemann surface G_∞ of function $\operatorname{Ln} z$.

$H(\rho)$ is the class of analytic in domain Δ_ρ functions $f(z)$. Let: $(0,l(\varphi)) = \{z; \arg z = \varphi, 0 < |z| < \infty\}$, $-\pi \le \varphi < \pi$. Let $\alpha \in [0,1)$, $\frac{1}{\rho} = 1-\alpha$, $\rho \ge 1$ and $f(z)$ be an arbitrary function of a complex variable $|f(re^{i\varphi})| \in L(0,l(\varphi))$, $|\varphi| \le \pi$, $0 < r < \infty$.

The function

$$D^{-\alpha} f(z) \equiv \frac{1}{\Gamma(\alpha)} \int_0^z (z-\xi)^{\alpha-1} f(\xi)d\xi, \qquad (2.1)$$

where the integration is made along the intercept connecting points 0 and z, $\arg(z-\xi)^{\alpha-1}=(\alpha-1)\arg z$, is called the Riemann–Liouville integral of order α of function $f(z)$, and the function

$$D^{1/\rho}f(z)\equiv\frac{d}{dz}D^{-\alpha}f(z) \tag{2.2}$$

is called the Riemann–Liouville derivative of order $1/\rho$ of function $f(z)$.

The operators

$$\begin{gathered} D^{0/\rho}f(z)\equiv f(z),\quad D^{1/\rho}f(z)\equiv\frac{d}{dz}D^{-\alpha}f(z),\\ D^{n/\rho}f(z)\equiv D^{\frac{1}{\rho}}D^{\frac{n-1}{\rho}}f(z),\qquad n\geq 2, \end{gathered} \tag{2.3}$$

are called Riemann–Liouville operators of successive differentiation of order n/ρ of function $f(z)$.

The Mittag–Leffler type function $E_\rho(z;\mu)=\sum\limits_{n=0}^{\infty}\dfrac{z^n}{\Gamma(\mu+n\rho^{-1})}$, $\rho>0$, is an entire function of order ρ with arbitrary value of parameter μ ([2], chap. VI, § 1).

For any $\mu>0,\alpha>0$ the following formula holds ([2], ch. III, formula (1.16)):

$$\begin{gathered} \frac{1}{\Gamma(\alpha)}\int_0^z(z-\xi)^{\alpha-1}E_\rho(\lambda\xi^{1/\rho};\mu)\xi^{\mu-1}d\xi=z^{\mu+\alpha+1}E_\rho(\lambda z^{1/\rho};\mu+\alpha),\\ z=re^{i\varphi},\ \xi=\tau e^{i\varphi},\ 0<\tau<r<l<+\infty,\ -\pi\leq\varphi<\pi. \end{gathered} \tag{2.4}$$

Lemma 2.1. Let $\rho\geq1$, λ, $\tilde\lambda$ be arbitrary complex parameters. The formula holds

$$\int_0^z e_\rho(z-\xi;\lambda)e_\rho(\xi;\tilde\lambda)d\xi=\frac{e_\rho(z;\lambda)-e_\rho(z;\tilde\lambda)}{\tilde\lambda-\lambda},\ z=re^{i\varphi},\ 0<r<l<+\infty,\ |\varphi|\leq\pi, \tag{2.5}$$

where $e_\rho(z;\lambda)\equiv E_\rho(-\lambda z^{1/\rho};1/\rho)z^{\frac{1}{\rho}-1}$, but the integration is along the segment connecting 0 and z points.

Note that for $z=x\in(0,+\infty)$ this Lemma was first established in more general form in [4] (see also [2], ch. III, (1.21)).

§ 3. Main Results. Let $\{\lambda_j\}_0^\infty$ be an arbitrary increasing sequence of positive numbers. Consider the sequence of operators on proper class of functions $f(z)$ for any $\rho\geq1$:

$$\begin{gathered} L^{0/\rho}f(z)\equiv f(z),\quad L^{n/\rho}f(z)\equiv\prod_{j=0}^{n-1}(D^{1/\rho}+\lambda_j)f(z),\\ n\geq1,\quad z=re^{i\varphi},\quad 0\leq r<l<+\infty,\quad |\varphi|\leq\pi, \end{gathered} \tag{3.1}$$

$$\tilde L^{n/\rho}f(z)\equiv D^{-\alpha}L^{n/\rho}f(z),\ \alpha\in[0;1),\ 1-\alpha=\frac{1}{\rho},\ n\geq0. \tag{3.2}$$

Then consider the following systems of functions $\left\{\varepsilon_n^{(\rho)}(z)\right\}_0^\infty$, $\left\{\Omega_n^{(\rho)}(z)\right\}_0^\infty$:

$$\varepsilon_0^{(\rho)}(z)\equiv e_\rho(z;\lambda_0),\ \ \varepsilon_n^{(\rho)}(z)\equiv\varepsilon^{(\rho)}(z;\lambda_0,\lambda_1,...,\lambda_n)=\sum_{k=0}^{n}C_k^{(n)}e_\rho(z;\lambda_k),\ \ z\in\Delta_\rho, \tag{3.3}$$

where $C_k^{(n)}=\left\{\prod_{j=0,\ j\neq k}^{n}(\lambda_j-\lambda_k)\right\}^{-1}$, $e_\rho(z;\lambda_k)\equiv E_\rho(-\lambda_k z^{1/\rho};1/\rho)z^{\frac{1}{\rho}-1}$.

$$\begin{gathered}\Omega_0^{(\rho)}(z)\equiv e_\rho(z;\lambda_0),\qquad \Omega_n^{(\rho)}(z)\equiv\Omega^{(\rho)}(z;\lambda_0,\lambda_1,...\lambda_n)\equiv\\ \equiv\int_0^z e_\rho(z-t_1;\lambda_0)dt_1\int_0^{t_1}e_\rho(t_1-t_2;\lambda_1)dt_2...\int_0^{t_{n-1}}e_\rho(t_{n-1}-t_n;\lambda_{n-1})e_\rho(t_n;\lambda_n)dt_n,\ n\geq 1,\end{gathered} \tag{3.4}$$

where $z=re^{i\varphi}$, $t_1=\tau_1e^{i\varphi},...,t_n=\tau_ne^{i\varphi}$, $0<\tau_n<\tau_{n-1}<...<\tau_1<r<l<+\infty$.

Note that for $z=x\in(0,+\infty)$ operators (3.1), (3.2) and systems of functions (3.3), (3.4) were first introduced in [1]. More general systems of operators and functions were introduced in [5], and for $\rho\in(0,1)$ similar operators and functions were introduced in [6].

Lemma 3.1. Let $\rho\geq 1$, $0\leq\alpha<1$. For any $n\geq 1$ the following formula holds:

$$\begin{gathered}\frac{1}{\lambda_{n+1}-\lambda_n}\left\{\varepsilon^{(\rho)}(z;\lambda_0,\lambda_1,...,\lambda_{n-1},\lambda_n)-\varepsilon^{(\rho)}(z;\lambda_0,\lambda_1,...,\lambda_{n-1},\lambda_{n+1})\right\}=\\ =\varepsilon^{(\rho)}(z;\lambda_0,\lambda_1,...,\lambda_n,\lambda_{n+1}),\ 0<\lambda_j<\lambda_{j+1},\ \ j=0,1,...,n+1,\ z\in\Delta_\rho.\end{gathered} \tag{3.5}$$

Lemma 3.2. The systems of functions $\left\{\varepsilon_n^{(\rho)}(z)\right\}_0^\infty$ and $\left\{\Omega_n^{(\rho)}(z)\right\}_0^\infty$ associated with the given sequence $\{\lambda_j\}_0^\infty$ are identical, i.e.

$$\varepsilon_n^{(\rho)}(z)\equiv\Omega_n^{(\rho)}(z),\ n\geq 0,\ z\in\Delta_\rho. \tag{3.6}$$

Proof. Note that according to (3.3), (3.4) we have for $n=0$

$$\varepsilon_0^{(\rho)}(z)=\Omega_0^{(\rho)}(z)\equiv e_\rho(z;\lambda_0)\equiv E_\rho(-\lambda_0 z^{1/\rho};1/\rho)z^{\frac{1}{\rho}-1}. \tag{3.7}$$

Let $n=1$. Then

$$\varepsilon_1^{(\rho)}(z)=\sum_{k=0}^{1}C_k^{(1)}e_\rho(z;\lambda_k)=\frac{1}{\lambda_1-\lambda_0}\{e_\rho(z;\lambda_0)-e_\rho(z;\lambda_1)\} \tag{3.8}$$

and

$$\Omega_1^{(\rho)}(z)=\int_0^z e_\rho(z-t;\lambda_0)e_\rho(t;\lambda_1)dt_1,\ \ z=re^{i\varphi},\ t=\tau e^{i\varphi}. \tag{3.9}$$

Using formula (2.9) we have

$$\Omega_1^{(\rho)}(z)=\frac{1}{\lambda_1-\lambda_0}\left\{e_\rho(z;\lambda_0)-e_\rho(z;\lambda_1)\right\}, \tag{3.10}$$

i.e. $\varepsilon_1^{(\rho)}(z)=\Omega_1^{(\rho)}(z)$.

Now availing of the methods of inductive reasoning, i.e. assuming that (3.6) holds for $n\geq 2$, we show that it is valid for $n+1$. According to (3.4), using formula (2.9) and Lemma 3.1 we have

$$\Omega_{n+1}^{(\rho)}(z)\equiv\int_0^z e_\rho(z-t_1;\lambda_0)dt_1\int_0^{t_1} e_\rho(t_1-t_2;\lambda_1)dt_2\ldots$$

$$\ldots\int_0^{t_{n-1}} e_\rho(t_{n-1}-t_n;\lambda_{n-1})\left\{\frac{e_\rho(t_n;\lambda_n)-e_\rho(t_n;\lambda_{n+1})}{\lambda_{n+1}-\lambda_n}\right\}dt_n=$$

$$=\frac{1}{\lambda_{n+1}-\lambda_n}\left(\int_0^z e_\rho(z-t;\lambda_0)dt_1\int_0^{t_1} e_\rho(t_1-t_2;\lambda_1)dt_2\ldots\int_0^{t_{n-1}} e_\rho(t_{n-1}-t_n;\lambda_{n-1})e_\rho(t_n;\lambda_n)dt_n-\right.$$

$$\left.-\int_0^z e_\rho(z-t_1;\lambda_0)dt_1\int_0^{t_1} e_\rho(t_1-t_2;\lambda_1)dt_2\ldots\int_0^{t_{n-1}} e_\rho(t_{n-1}-t_n;\lambda_{n-1})e_\rho(t_n;\lambda_{n+1})dt_n\right)=$$

$$=\frac{1}{\lambda_{n+1}-\lambda_n}\left(\varepsilon^{(\rho)}\left(z;\lambda_0,\lambda_1,\ldots,\lambda_{n-1},\lambda_n\right)-\varepsilon^{(\rho)}\left(z;\lambda_0,\lambda_1,\ldots,\lambda_{n-1},\lambda_{n+1}\right)\right)=$$

$$=\varepsilon^{(\rho)}(z;\lambda_0,\lambda_1,\ldots,\lambda_{n-1},\lambda_n,\lambda_{n+1}).$$

Lemma 3.2 is proved.

Lemma 3.3. Let $\rho\ge 1$, $f(z)\in H(\rho)$, $f(re^{i\varphi})\in L(0;l(\varphi))$, $z\in\Delta_\rho$. Then for any $n\ge 1$ the formula holds

$$\int_0^z e_\rho(z-t_1;\lambda_0)dt_1\int_0^{t_1} e_\rho(t_1-t_2;\lambda_1)dt_2\ldots\int_0^{t_{n-1}} e_\rho(t_{n-1}-t_n;\lambda_{n-1})dt_n\times$$
$$\times\int_0^{t_n} e_\rho(t_n-t_{n+1};\lambda_n)f(t_{n+1})dt_{n+1}=\int_0^z \varepsilon_n^{(\rho)}(z-\tau)f(\tau)d\tau, \tag{3.11}$$

where $z=re^{i\varphi}$, $t_1=\tau_1 e^{i\varphi},\ldots,t_{n+1}=\tau_{n+1}e^{i\varphi}$, $0<\tau_{n+1}<\tau_n<\ldots<\tau_1<r<l<+\infty$. Lemma 3.3 is proved in the same way as the proof of Lemma 4.1 for $z=x\in(0,+\infty)$ [1].

Lemma 3.4. Let $\rho\ge 1, f(z)\in H(\rho)$, $f(re^{i\varphi})\in L(0,l(\varphi))$. Then the function

$$y(z;\{\lambda_j\}_0^n)\equiv\int_0^z \varepsilon_n^{(\rho)}(z-t)f(t)dt,\ z=re^{i\varphi},\ t=\tau e^{i\varphi}, \tag{3.12}$$

is the solution of the following Cauchy type problem:

$$L^{\frac{n+1}{\rho}}y(z)=f(z), \tag{3.13}$$

$$\tilde{L}^{k/\rho}y(z)|_{z=0}=0,\quad k=0,1,\ldots,n. \tag{3.14}$$

The proof of Lemma 3.4 is not given here, because similar problems for $z=x\in(0,+\infty)$ have been discussed in [1] (see Lemma 3.2).

Lemma 3.5.

1^0. For any $n\ge 0$ the following relations hold:

$$D^{-\alpha}\{L^{k/\rho}[\varepsilon_n^{(\rho)}(z)]\}\equiv L^{k/\rho}[\varepsilon_n^{(\rho)}(z)]\equiv 0,\ k\ge n+1,\ \rho\ge 1, z\in\Delta_\rho, \tag{3.15}$$

$$D^{-\alpha}\{L^{n/\rho}[\varepsilon_n^{(\rho)}(z)]\}\equiv E_\rho(-\lambda_n z^{1/\rho};1),\ z\in\Delta_\rho. \tag{3.16}$$

2^0. For any $n\ge 1$

$$\lim_{|z|\to 0} D^{-\alpha}\{L^{k/\rho}[\varepsilon_n^{(\rho)}(z)]\}=0,\ 0\le k\le n-1,\ z\in\Delta_\rho. \tag{3.17}$$

Proof.

1^0. Let $k=n+1$, then according to (3.1), (3.5) we have

$$\overset{\frac{n+1}{\rho}}{L}[\varepsilon_n^{(\rho)}(z)] = \overset{\frac{n+1}{\rho}}{L}\left(\sum_{k=0}^{n} C_k^{(n)} e_\rho(z;\lambda_k)\right) = \sum_{k=0}^{n} C_k^{(n)} \prod_{j=0}^{n}(D^{1/\rho}+\lambda_j)e_\rho(z;\lambda_k). \quad (3.18)$$

But since $(D^{1/\rho}+\lambda)e_\rho(z;\lambda)\equiv 0$ (see [7]), then from (3.18) we get

$$\overset{\frac{n+1}{\rho}}{L}[\varepsilon_n^{(\rho)}(z)] = \sum_{k=0}^{n} C_k^{(n)} \prod_{j=0}^{n}(\lambda_j-\lambda_k)e_\rho(z;\lambda_k). \quad (3.19)$$

Since $\prod_{j=0}^{n}(\lambda_j-\lambda_k)=0$ for $k=0,1,...,n$, it follows from (3.19) that $\overset{\frac{n+1}{\rho}}{L}[\varepsilon_n^{(\rho)}(z)]\equiv 0$. It is easy to see under the assumption of $k\ge n+2$ that

$$L^{k/\rho}\left[\varepsilon_n^{(\rho)}(z)\right] = \prod_{j=n+1}^{k-1}(D^{1/\rho}+\lambda_j)\prod_{j=0}^{n}(D^{1/\rho}+\lambda_j)\left[\varepsilon_n^{(\rho)}(z)\right] =$$

$$= \prod_{j=n+1}^{k-1}(D^{1/\rho}+\lambda_j)\overset{\frac{n+1}{\rho}}{L}\left[\varepsilon_n^{(\rho)}(z)\right]\equiv 0.$$

Now note that $\tilde{L}^{k/\rho}\left[\varepsilon_n^{(\rho)}(z)\right]\equiv D^{-\alpha}L^{k/\rho}\left[\varepsilon_n^{(\rho)}(z)\right]\equiv 0,\quad k\ge n+1.$ Further we have

$$L^{n/\rho}\left[\varepsilon_n^{(\rho)}(z)\right] = \sum_{k=0}^{n} C_k^{(n)}\prod_{j=0}^{n-1}(D^{1/\rho}+\lambda_j)e_\rho(z;\lambda_k) = \sum_{k=0}^{n} C_k^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_k)e_\rho(z;\lambda_k) =$$
$$= C_n^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_n)e_\rho(z;\lambda_n), \quad (3.20)$$

but since $C_n^{(n)}\prod_{j=0}^{n-1}(\lambda_j-\lambda_n) = \left\{\prod_{j=0}^{n-1}(\lambda_j-\lambda_n)\right\}^{-1}\prod_{j=0}^{n-1}(\lambda_j-\lambda_n)=1,$ then from (3.20) it follows that

$$L^{n/\rho}\left[\varepsilon_n^{(\rho)}(z)\right] = e_\rho(z;\lambda_n)\equiv E_\rho(-\lambda_n z^{1/\rho};1/\rho)z^{\frac{1}{\rho}-1}. \quad (3.21)$$

Using formula (2.4) we obtain from (3.21)

$$D^{-\alpha}L^{n/\rho}\left[\varepsilon_n^{(\rho)}(z)\right]\equiv \tilde{L}^{n/\rho}\left[\varepsilon_n^{(\rho)}(z)\right] = E_\rho(-\lambda_n z^{1/\rho};1).$$

2^0. Let now $n\ge 1,\ \ 0\le k\le n-1$. Then we have:

$$D^{-\alpha}\left\{L^{k/\rho}\left[\varepsilon_n^{(\rho)}(z)\right]\right\} = D^{-\alpha}\left(\sum_{i=0}^{n} C_i^{(n)}\prod_{j=0}^{k-1}(D^{1/\rho}+\lambda_j)e_\rho(z;\lambda_i)\right) =$$

$$= D^{-\alpha}\left(\sum_{i=k}^{n} C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)e_\rho(z;\lambda_i)\right) = \sum_{i=k}^{n} C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)D^{-\alpha}\{e_\rho(z;\lambda_i)\} = \quad (3.22)$$

$$= \sum_{i=k}^{n} C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)E_\rho(-\lambda_i z^{1/\rho};1).$$

Note that $E_\rho(-\lambda_i z^{1/\rho};1)\,|_{z=0}=0$, and

$$C_i^{(n)}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)=\left\{\prod_{j=0,\,j\neq i}^{n}(\lambda_j-\lambda_i)\right\}^{-1}\prod_{j=0}^{k-1}(\lambda_j-\lambda_i)=\left\{\prod_{j=k,\,j\neq i}^{n}(\lambda_j-\lambda_i)\right\}^{-1},$$

it follows from (3.22)

$$\lim_{|z|\to 0}D^{-\alpha}\{L^{k/\rho}[\varepsilon_n^{(\rho)}(z)]\}=\sum_{i=k}^{n}\left\{\prod_{j=k,\,j\neq i}^{n}(\lambda_j-\lambda_i)\right\}^{-1}=0, \quad \text{(see [1], (3.31)).}$$

Lemma 3.5 is proved.

Lemma 3.6. For any $n\geq 0$ in a sum of the form

$$P_n(z)=\sum_{k=0}^{n}a_k\varepsilon_k^{(\rho)}(z),\ z\in\Delta_\rho, \tag{3.23}$$

the coefficients $\{a_k\}_0^n$ may be determined from formulae

$$a_j=\tilde{L}^{j/\rho}P_n(0), \qquad j=0,1,\dots,n. \tag{3.24}$$

Proof. Let $0\leq j\leq n-1$. Applying the operator $\tilde{L}^{j/\rho}$ to function $P_n(z)$ and using (3.15)–(3.17), we obtain

$$\begin{aligned}\tilde{L}^{j/\rho}P_n(z)&=\sum_{k=0}^{j-1}a_k\tilde{L}^{j/\rho}\{\varepsilon_k^{(\rho)}(z)\}+a_j\tilde{L}^{j/\rho}\{\varepsilon_j^{(\rho)}(x)\}+\\&+\sum_{k=j+1}^{n}a_k\tilde{L}^{j/\rho}\{\varepsilon_k^{(\rho)}(z)\}=a_jE_\rho(-\lambda_jz^{1/\rho};1)+\sum_{k=j+1}^{n}a_k\tilde{L}^{j/\rho}\{\varepsilon_k^{(\rho)}(z)\}.\end{aligned} \tag{3.25}$$

But since $E_\rho(0;1)=1$, then from (3.25) and (3.17) we obtain $\lim_{|z|\to 0}\tilde{L}^{j/\rho}P_n(z)\equiv\tilde{L}^{j/\rho}P_n(0)=a_j,\ \ 0\leq j\leq n-1$.

Then the operator $\tilde{L}^{n/\rho}$ is applied to the function $P_n(z)$:

$$\tilde{L}^{n/\rho}P_n(z)=\sum_{k=0}^{n}a_k\tilde{L}^{n/\rho}\{\varepsilon_k^{(\rho)}(z)\}=a_n\tilde{L}^{n/\rho}\{\varepsilon_n^{(\rho)}(z)\}=a_nE_\rho(-\lambda_nz^{1/\rho};1). \tag{3.26}$$

From here we obtain $a_n=\lim_{|z|\to 0}\tilde{L}^{n/\rho}P_n(z)\equiv\tilde{L}^{n/\rho}P_n(0)$. Lemma 3.6 is proved.

Now denote by $C_{n+1}^{(\rho)}[0;l(\varphi)],\ 0<l<+\infty,\ \rho\geq 1$, the set of functions $f(z)\in H(\rho)$, satisfying the following conditions:

1) the functions $\tilde{D}^{k/\rho}f(z)=D^{-\alpha}D^{k/\rho}f(z),\quad k=0,1,\dots,n$, are continuous on $[0;l(\varphi))$;

2) the function $D^{(n+1)/\rho}f(z)$ is continuous on $(0;l(\varphi))$ and belongs to $L(0;l(\varphi))$.

It is easy to see that according to (3.1), (3.2) this functions $\tilde{L}^{k/\rho}f(z)=D^{-\alpha}L^{k/\rho}f(z),\ \ k=0,1,\dots,n$, are continuous on $[0;l(\varphi))$ and function $L^{(n+1)/\rho}f(z)$ is continuous on $[0;l(\varphi))$ and belongs to $L(0;l(\varphi))$.

Theorem 3.1. If $f(z)\in C_{n+1}^{(\rho)}[0;l(\varphi)]$, then for any $n>\rho-1$

$$f(z)=\sum_{k=0}^{n}\tilde{L}^{k/\rho}f(0)\varepsilon_k^{(\rho)}(z)+R_n(z;f),\ \ z\in\Delta_\rho\cap(0;l(\varphi)), \tag{3.27}$$

where

$$R_n(z;f)=\int_0^z \varepsilon_n^{(\rho)}(z-t)L^{(n+1)/\rho}\{f(t)\}dt,\quad z=re^{i\varphi},\ t=\tau e^{i\varphi},\ 0<\tau<r<l<+\infty. \quad (3.28)$$

Proof. Let

$$P_n(z;f)=\sum_{k=0}^{n}\tilde{L}^{k/\rho}f(0)\varepsilon_k^{(\rho)}(z). \quad (3.29)$$

We put

$$f(z)-P_n(z;f)\equiv R_n(z;f). \quad (3.30)$$

It is easy to see that according to Lemma 3.6 $\tilde{L}^{k/\rho}P_n(0;f)=\tilde{L}^{k/\rho}f(0)$, $k=0,1,2,...,n$, and consequently $\tilde{L}^{k/\rho}\left[R_n(z;f)\right]\Big|_{z=0}=0,\ k=0,1,...,n$. Further, since according to (3.15) $L^{(n+1)/\rho}[\varepsilon_k^{(\rho)}(z)]\Big|_{z=0}=0,\ k=0,1,2,...,n,\ z\in\Delta_\rho$, then $L^{(n+1)/\rho}[R_n(z;f)]\equiv L^{(n+1)/\rho}f(z)$. Now consider the problem:

$$L^{(n+1)/\rho}\left[R_n\left(z;f\right)\right]\equiv L^{(n+1)/\rho}f(z), \quad (3.31)$$

$$\tilde{L}^{k/\rho}\left[R_n\left(z;f\right)\right]\Big|_{z=0}=0,\ k=0,1,...,n. \quad (3.32)$$

According to Lemma 3.4 this problem (3.31)–(3.32) has a unique solution $R_n(z;f)=\int_0^z \varepsilon_n^{(\rho)}(z-t)L^{(n+1)/\rho}f(t)dt$. It follows from formulae (3.28)–(3.30) that

$$f(z)=\sum_{k=0}^{n}\tilde{L}^{k/\rho}f(0)\varepsilon_n^{(\rho)}(z)+\int_0^z \varepsilon_n^{(\rho)}(z-t)L^{(n+1)/\rho}f(t)dt.$$

Theorem 3.1 is proved.

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REFERENCES

1. **Dzhrbashyan M.M. and Sahakyan B.A.** Izv. AN SSSR. Matematica., 1975, v. 39, № 1, p. 69–122 (in Russian).
2. **Dzhrbashyan M.M.** Integral Transforms and Representations of Functions in the Complex Domain. M.: Nauka, 1966 (in Russian).
3. **Dzhrbashyan M.M.** Izv. AN Arm. SSR. Matematica, 1968, v. 3, № 3, p. 171–248 (in Russian).
4. **Dzhrbashyan M.M. and Nersesyan A.B.** Izv. AN Arm. SSR. Phyzmat. nauki, 1959, v. 12, № 5, p. 17–42 (in Russian).
5. **Dzhrbashyan M.M. and Sahakyan B.A.** Izv. AN Arm. SSR. Matematica, 1977, v. XII, № 1, p. 66–83 (in Russian).
6. **Sahakyan B.A.** Uchenye zapiski EGU, 1988, № 3, p. 30–40 (in Russian).
7. **Sahakyan B.A.** Proceedings of the YSU. Physical and Mathematical Sciences, 2010, № 3, p. 29–34.