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Mathematics

ON COMPACTNESS OF A CLASS OF FIRST ORDER LINEAR DIFFERENTIAL OPERATORS

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In the present article a class of first order linear differential operators with unbounded coefficients is investigated. The compactness of operators is proved.

***Keywords*:** bounded differential operator, compact differential operator, first order differential operator.

Let $Q \subset R^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial Q \in C^1$. Consider the first order differential expression

$$Tu \equiv (\bar{b}(x), \nabla u(x)) - \operatorname{div}(\bar{c}(x)u(x)) + d(x)u(x), \quad u \in \overset{\circ}{W}{}_2^1(Q),$$

with coefficients $\bar{b}(x) = (b^{(1)}(x), \ldots, b^{(n)}(x))$, $\bar{c}(x) = (c^{(1)}(x), \ldots, c^{(n)}(x))$ and $d(x)$ that are measurable and bounded on each strong inner subdomain of the domain Q.

For an arbitrary $u, v \in \overset{\circ}{W}{}_2^1(Q)$ define

$$\langle Tu, v \rangle \equiv \int_Q ((\bar{b}(x), \nabla u(x))v(x) + (\bar{c}(x)u(x), \nabla v(x)) + d(x)u(x)v(x))dx, \quad v \in \overset{\circ}{W}{}_2^1(Q).$$

Assume that the coefficients $\bar{b}(x)$, $\bar{c}(x)$ and $d(x)$ satisfy the conditions

$$\left|\bar{b}(x)\right| = O\left(\frac{1}{r(x)}\right) \text{ as } r(x) \to 0,$$

where $r(x)$ is the distance of a point $x \in Q$ from the boundary ∂Q,

$$\int_0 tC^2(t)dt < \infty \quad \text{with} \quad C(t) = \sup_{r(x) \geq t} \left|\bar{c}(x)\right|,$$

$$\int_0 t^3 D^2(t)dt < \infty \quad \text{with} \quad D(t) = \sup_{r(x) \geq t} \left|d(x)\right|.$$

In [1] it was shown that T is a bounded linear operator from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$. The aim of this article is to obtain conditions on coefficients $\bar{b}(x), \bar{c}(x)$

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and $d(x)$, for which T is a linear compact operator from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$. This property has important applications in studying the solvability of the problems of mathematical physics, see, for example, [2, 3].
We prove the below theorem.

Theorem. Let the below conditions hold

$$\left|\bar{b}(x)\right| = o\left(\frac{1}{r(x)}\right) \text{ as } r(x) \to 0, \tag{1}$$

and there exist monotone functions $\omega_i(t) \to 0$, as $t \to +0$, $i = 1,2$, such that

$$\int_0 \frac{tC^2(t)}{\omega_1(t)} dt < \infty, \text{ where } C(t) = \sup_{r(x)\geq t} \left|\bar{c}(x)\right|, \tag{2}$$

$$\int_0 \frac{t^3 D^2(t)}{\omega_2(t)} dt < \infty, \text{ where } D(t) = \sup_{r(x)\geq t} \left|d(x)\right|. \tag{3}$$

Then the operator T is a compact linear operator from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$.

Proof of Theorem. We shall follow the scheme of proof of the theorem from [1].

Let $x^0 \in \partial Q$ be an arbitrary point of the boundary ∂Q of the domain Q and (x', x_n) be a local coordinate system with the origin x^0 and the x_n axis directed along the inner normal $\nu(x^0)$ to ∂Q at the point x^0. Since $\partial Q \in C^1$, there exists a positive number $r_{x^0} > 0$ and a function $\varphi_{x^0} \in C^1(R^{n-1})$ with properties

$$\varphi_{x^0}(0) = 0, \ \nabla\varphi_{x^0}(0) = 0 \text{ and } \left|\nabla\varphi_{x^0}(x')\right| \leq \frac{1}{2} \text{ for all } x' \in R^{n-1},$$

such that the intersection of the domain Q with the ball $U_{x^0}^{(r_{x^0})} = \{x : |x - x^0| < r_{x^0}\}$ of radius r_{x^0} and the centre x^0 has the form $Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n > \varphi_{x^0}(x')\}$.
Then $\partial Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n = \varphi_{x^0}(x')\}$. Let $l_{x_0} = \frac{r_{x_0}}{\sqrt{2}}$. From the covering $\{U_{x^0}^{(l_{x^0})}, x^0 \in \partial Q\}$ of the boundary ∂Q select a finite subcovering $U_{x^m}^{(l_{x^m})}$, $m = 1,...,p$. Denote for simplicity $U_{x^m}^{(l_{x^m})}$ by U_m, r_{x^m} by r_m, l_{x^m} by l_m, φ_{x^m} by φ_m, $m = 1,...,p$. Set $h = \frac{1}{3}\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)\min(r_1,...,r_p)$. Then each of the curvilinear "cylinders" $\prod_m^{l_m,h} = \{(x', x_n) : |x'| < l_m, \varphi_m(x') < x_n < \varphi_m(x') + h\}$, $m = 1,...,p$, is contained in the corresponding ball U_m, and also by $U_m \cap Q$ (recall that (x', x_n) are the coordinates of a point in a local system of coordinates with origin at x^m).
Let $l_0 < h$ be such a positive number that the complement of the domain $Q_{l_0} = \{x \in Q : r(x) = \text{dist}(x, \partial Q) > l_0\}$ in Q is contained in the union of the "cylinders" $\prod_m^{l_m,h}$, $m = 1,...,p$, i.e. $Q^{l_0} = \{x \in Q : r(x) = \text{dist}(x, \partial Q) \leq l_0\} \subset \bigcup_{m=1}^{p} \prod_m^{l_m,h}$.

Easily verified that for all $x=(x',x_n)\in\Pi_m^{l_m,h}$, $m=1,\dots,p$,

$$r(x)\le x_n-\varphi_m(x')\le\frac{\sqrt{5}}{2}r(x).$$

We fix an index m, $1\le m\le p$, and take a local coordinate system with origin at x^m.

Now define mappings L and L_{-1} of the space R^n onto itself by relations $L(x)=(x',x_n-\varphi_m(x'))$, where $x=(x',x_n)$ and $L_{-1}(y)=(y',y_n+\varphi_m(y'))$ with $y=(y',y_n)$. The image of $\prod_m^{l_m,h}$ under the mapping L will be denoted by $\tilde{\prod}_m^{l_m,h}$: $L(\tilde{\prod}_m^{l_m,h})=\tilde{\prod}_m^{l_m,h}$.

Consider the sequence of operators

$$T_k u=(\bar{b}_k(x),\nabla u(x))-\operatorname{div}(\bar{c}_k(x)u(x))+d_k(x)u(x),\quad u\in\overset{\circ}{W}{}_2^1(Q),\quad k=1,2,\dots$$

$$\bar{b}_k(x)=\begin{cases}\bar{b}(x), & \text{if } r(x)>\dfrac{1}{k},\\ 0, & \text{if } r(x)\le\dfrac{1}{k},\end{cases}\quad \bar{c}_k(x)=\begin{cases}\bar{c}(x), & \text{if } r(x)>\dfrac{1}{k},\\ 0, & \text{if } r(x)\le\dfrac{1}{k},\end{cases}\quad d_k(x)=\begin{cases}d(x), & \text{if } r(x)>\dfrac{1}{k},\\ 0, & \text{if } r(x)\le\dfrac{1}{k}.\end{cases}$$

It can be readily verified that the operator T_k is a compact linear operator from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$. Indeed, let $\{w(x)\}$ be a bounded set in $\overset{\circ}{W}{}_2^1(Q)$. Then sets $\{(\bar{b}_k(x),\nabla w(x))\}$, $\{\bar{c}_k(x)w(x)\}$ and $\{d_k(x)w(x)\}$ are bounded in $L_2(Q)$ and by that are compact in $\overset{\circ}{W}{}_2^{-1}(Q)$ (see, for example, [4]). Hence, for the proof of the theorem it is sufficient to show that $\|T-T_k\|\to 0$ as $k\to\infty$.

Without loss of generality we suggest that $k>\dfrac{1}{l_0}$ and functions $\omega_1(t)$, $\omega_2(t)$ are positive. In view of (1) there exists a monotone function $\varepsilon(t)\to 0$, as $t\to+0$, such that $\left|\bar{b}(x)\right|\le\dfrac{\varepsilon(r(x))}{r(x)}$. For $u\in\overset{\circ}{W}{}_2^1(Q)$ and $\eta\in C_0^\infty(Q)$ consider

$$\langle (T-T_k)u,\eta\rangle=\int_{Q^{1/k}}\Big((\bar{b}(x),\nabla u(x))\eta(x)+(\bar{c}(x)u(x),\nabla\eta(x))+d(x)u(x)\eta(x)\Big)dx.$$

Denote $u(y',y_n+\varphi(y'))=\tilde{u}(y)$, $\eta(y',y_n+\varphi(y'))=\tilde{\eta}(y)$.

Due to (1), (2) and (3), we have

$$\left|\langle (T-T_k)u,\eta\rangle\right|\le\int_{Q^{1/k}}\left(\frac{\varepsilon(r(x))|\nabla u(x)||\eta(x)|}{r(x)}+C(r(x))|u(x)||\nabla\eta(x)|+D(r(x))|u(x)||\eta(x)|\right)dx\le$$

$$\le\varepsilon\left(\frac{1}{k}\right)\int_{Q^{1/k}}\frac{|\nabla u(x)||\eta(x)|}{r(x)}dx+\omega_1^{1/2}\left(\frac{1}{k}\right)\int_{Q^{1/k}}\frac{C(r(x))}{\omega_1^{1/2}(r(x))}|u(x)||\nabla\eta(x)|dx+\qquad(4)$$

$$+\omega_2^{1/2}\left(\frac{1}{k}\right)\int_{Q^{1/k}}\frac{D(r(x))}{\omega_2^{1/2}(r(x))}|u(x)||\eta(x)|dx.$$

Let us estimate

$$I_1 = \int_{Q^{1/k}} \frac{|\nabla u(x)||\eta(x)|}{r(x)} dx \le \int_{Q^{l_0}} \frac{|\nabla u(x)||\eta(x)|}{r(x)} dx \le \sum_{m=1}^{p} \int_{\Pi_m^{l_m,h}} \frac{|\nabla u(x)||\eta(x)|}{r(x)} dx .$$

In view of the Hardy inequality (see, for example, [5]) for $m = 1,\dots,p$ the following estimate holds:

$$\int_{\Pi_m^{l_m,h}} \frac{|\nabla u(x)||\eta(x)|}{r(x)} dx \le \sqrt{\frac{5}{2}} \int_{\tilde{\Pi}_m^{l_m,h}} \frac{|\nabla \tilde{u}(y)||\tilde{\eta}(y)|}{y_n} dy \le \sqrt{\frac{5}{2}} \left(\int_{\tilde{\Pi}_m^{l_m,h}} |\nabla \tilde{u}(y)|^2 dy \right)^{1/2} \left(\int_{\tilde{\Pi}_m^{l_m,h}} \frac{\tilde{\eta}^2(y)}{y_n^2} dy \right)^{1/2} \le$$

$$\le \sqrt{5} \left(\int_{\Pi_m^{l_m,h}} |\nabla u(x)|^2 dx \right)^{1/2} \left(\int_{\tilde{\Pi}_m^{l_m,h}} \frac{\tilde{\eta}^2(y)}{y_n^2} dy \right)^{1/2} \le \mathrm{const} \|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} .$$

Thus,

$$I_1 \le \mathrm{const} \|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} , \tag{5}$$

where the constant does not depend on u and η .

Next $I_2 = \int_{Q^{1/k}} \frac{C(r(x))}{\omega_1^{\frac{1}{2}}(r(x))} |u(x)||\nabla \eta(x)| dx \le \int_{Q^{l_0}} \frac{C(r(x))}{\omega_1^{1/2}(r(x))} |u(x)||\nabla \eta(x)| dx \le$

$$\le \sum_{m=1}^{p} \int_{\Pi_m^{l_m,h}} \frac{C(r(x))}{\omega_1^{1/2}(r(x))} |u(x)||\nabla \eta(x)| dx .$$

For $m = 1,\dots,p$ we have

$$\int_{\Pi_m^{l_m,h}} \frac{C(r(x))}{\omega_1^{1/2}(r(x))} |u(x)||\nabla \eta(x)| dx \le \left(\int_{\Pi_m^{l_m,h}} \frac{C^2(r(x))}{\omega_1(r(x))} u^2(x) dx \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \left(\int_{\tilde{\Pi}_m^{l_m,h}} \frac{C^2\left(\frac{2}{\sqrt{5}} y_n\right)}{\omega_1\left(\frac{2}{\sqrt{5}} y_n\right)} \tilde{u}^2(y) dy \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le \left(\int_{\tilde{\Pi}_m^{l_m,h}} \frac{C^2\left(\frac{2}{\sqrt{5}} y_n\right)}{\omega_1\left(\frac{2}{\sqrt{5}} y_n\right)} y_n \int_0^{y_n} |\nabla \tilde{u}(y',\tau)|^2 d\tau dy \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \left(\int_0^h dy_n \frac{C^2\left(\frac{2}{\sqrt{5}} y_n\right)}{\omega_1\left(\frac{2}{\sqrt{5}} y_n\right)} y_n \int_{|y'|<l_m} dy' \int_0^h d\tau |\nabla \tilde{u}(y',\tau)|^2 \right)^{1/2} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} \le$$

$$\le \sqrt{2} \left(\int_0^h \frac{C^2\left(\frac{2}{\sqrt{5}} y_n\right)}{\omega_1\left(\frac{2}{\sqrt{5}} y_n\right)} y_n dy_n \right)^{1/2} \|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} .$$

Thus, we get

$$I_2 \le \mathrm{const} \|u\|_{\overset{\circ}{W}{}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}{}_2^1(Q)} , \tag{6}$$

where the constant does not depend on u and η .

Similarly we obtain

$$I_3=\int_{Q^{1/k}}\frac{D(r(x))}{\omega_2^{1/2}(r(x))}|u(x)||\eta(x)|dx\le\int_{Q^0}\frac{D(r(x))}{\omega_2^{1/2}(r(x))}|u(x)||\eta(x)|dx\le\sum_{m=1}^{p}\int_{\Pi_m^{l_m,h}}\frac{D(r(x))}{\omega_2^{1/2}(r(x))}|u(x)||\eta(x)|dx.$$

Finally, for $m=1,...,p$ we get

$$\int_{\Pi_m^{l_m,h}}\frac{D(r(x))}{\omega_2^{1/2}(r(x))}|u(x)||\eta(x)|dx\le\int_{\tilde\Pi_m^{l_m,h}}\frac{D\left(\frac{2}{\sqrt5}y_n\right)}{\omega_2^{1/2}\left(\frac{2}{\sqrt5}y_n\right)}|\tilde u(y)||\tilde\eta(y)|dy\le$$

$$\le\left(\int_{\tilde\Pi_m^{l_m,h}}\frac{D^2\left(\frac{2}{\sqrt5}y_n\right)}{\omega_2\left(\frac{2}{\sqrt5}y_n\right)}y_n^2\tilde u^2(y)dy\right)^{1/2}\left(\int_{\tilde\Pi_m^{l_m,h}}\frac{\tilde\eta^2(y)}{y_n^2}dy\right)^{1/2}\le$$

$$\le\text{const}\left(\int_{\tilde\Pi_m^{l_m,h}}\frac{D^2\left(\frac{2}{\sqrt5}y_n\right)}{\omega_2\left(\frac{2}{\sqrt5}y_n\right)}y_n^3\int_0^{y_n}|\nabla\tilde u(y',\tau)|^2d\tau dy\right)^{1/2}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}\le$$

$$\le\text{const}\left(\int_0^h\frac{D^2\left(\frac{2}{\sqrt5}y_n\right)}{\omega_2\left(\frac{2}{\sqrt5}y_n\right)}y_n^3dy_n\right)^{1/2}\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)}.$$

Thus,

$$I_3\le\text{const}\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)},\qquad(7)$$

where the constant does not depend on u and η. From (4)–(7) we obtain the estimate

$$|\langle(T-T_k)u,\eta\rangle|\le\text{const}\left(\varepsilon\left(\frac1k\right)+\omega_1^{1/2}\left(\frac1k\right)+\omega_2^{1/2}\left(\frac1k\right)\right)\|u\|_{\overset{\circ}{W}{}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}{}_2^1(Q)},$$

where the constant does not depend on u and η.

From this it immediately follows that $\|T-T_k\|\le\text{const}\left(\varepsilon\left(\frac1k\right)+\omega_1^{1/2}\left(\frac1k\right)+\omega_2^{1/2}\left(\frac1k\right)\right)$

and consequently $\|T-T_k\|\to0$ as $k\to\infty$. The Theorem is proved.

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R E F E R E N C E S

1. **Dumanyan V.Zh.** Proceedings of the YSU. Physical and Mathematical Sciences, 2011, № 1, p. 3–6.
2. **Ladyzhenskaya O.A., Uraltseva N.N.** Linear and Quasilinear Elliptic Equations. Academic Press, 1968.
3. **Mikhailov V.P.** Partial Differential Equations. 1st ed. M.: Nauka, 1976; English transl. of 1st ed. M.: Mir, 1978.
4. **Mikhailov V.P., Gushchin A.K.** Advanced Topics of "Equations of Mathematical Physics". Lecture courses of SEC. M., Steklov Mathematical Institute of RAS, 2007.
5. **Maz'ya V.G.** Sobolev Spaces. L.: Leningrad Univ. Press, 1985 (in Russian); English transl. from the Russian by T.O. Shaposhnikova. Berlin–New York: Springer-Verlag, 1985.