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ON THE OPTIMAL STABILIZATION OF A DOUBLE MATHEMATICAL PENDULUM HAVING A MOVABLE SUSPENSION CENTER

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The problem of optimal stabilizatioin of a double pendulum, when its suspension center moved in the horizontal direction according to the given law, has been treated. The problem was reduced to the case of a linear nonuniform system that was solved in the event when the first and the second pendulums had equal masses and lengths. An optimal Lyapunov function and an optimal control action have been constructed.

Keywords: driven double pendulum, optimal stabilization, equations of motion, Lagrange's equations, Lyapunov function, Lyapunov–Bellman method.

Introduction. The problems of stabilization and control of a mathematical pendulum having a movable suspension center were the subjects of study of many researchers at different times. As early in 19th century, English researcher J. Reyleigh [1] has shown, that the lower stable equilibrium position of a pendulum might become unstable, when its suspension center oscillated at a specific frequency. An English mathematician In Stephenson proved [2], that it was possible to keep the pendulum at the upper equilibrium position, if its suspension center is oscillated in the vertical direction. Then, a comprehensive study of this problem was conducted by P.L. Kapitsa [3]. The investigation of driven mathematical pendulum is of importance also nowadays, as was dealt with in [4–6]. In [4] the problem of double pendulum stability has been studied for four possible equilibrium positions, the suspension point of the first pendulum being oscillated at high frequency in the vertical direction. A comprehensive analysis of studies concerning the problems of stability, control and stabilization of driven pendulum is given in [6]. The stabilization problem of a driven single pendulum, when the suspension center moved in the horizontal direction, was studied in [7].

In the present paper the optimal stabilization problem of a driven double pendulum is considered, when the upper suspension center is moved in the horizontal direction according to a given law.

Problem Statement and Equations of Motion. Suppose there is a double pendulum composed of l_1 and l_2 long lightweight rods holding material particles M_1 and M_2 of m_1 and m_2 masses respectively. Now assume, that upper pendulum may swing about the horizontal axis Oz (that is perpendicular to the drawing plane and is

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not seen in the drawing), and the lower pendulum swings about an axis passing trough the point M_1 perpendicular to plane xOy. Ox axis has a horizontal direction, whereas Oy axis is directed downwards normal to that (Fig. 1).

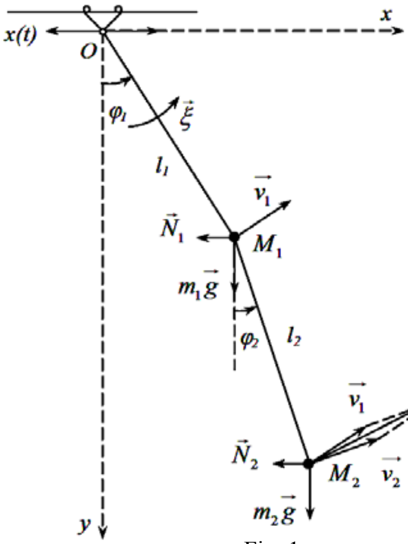

Fig. 1.

If we assume, that the upper suspension center O moves in the horizontal direction according to the given law, then the gravity forces $m_1 g$, $m_2 g$ and translational inertia loads $N_1 = m_1\ddot{x}$ and $N_2 = m_2\ddot{x}$ will act on the material particles M_1 and M_2 respectively.

Assume also, that the device at the suspension center acts on OM_1 rod of pendulum with controlling moment $\vec{\xi}$.

The coordinates of system will be uniquely determined by angles φ_1, φ_2 made by the vertical and rods l_1, l_2 respectively. The equations of motion of the system are derived from the Lagrange equations [8]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}_i}\right) - \frac{\partial T}{\partial \varphi_i} = Q_i - \frac{\partial \Pi}{\partial \varphi_i}, \qquad i = 1,2\,, \tag{1}$$

where T and Π are the kinetic and potential energies respectively, and Q_i are generalized forces. The velocity of point M_1 $\vec{v}_1$ is equal to $v_1 = l_1\dot{\varphi}_1$, and the square of the velocity of point M_2 is written as

$$v^2 = v_1^2 + v_2^2 + 2v_1v_2\cos\alpha = l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2)\,.$$

The kinetic and potential energies can be written as:

$$T = \frac{m_1 l_1^2}{2}\dot{\varphi}_1^2 + \frac{m_2}{2}(l_1^2\dot{\varphi}_1^2 + l_2^2\dot{\varphi}_2^2 + 2l_1l_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1 - \varphi_2))\,, \tag{2}$$

$$\Pi = m_1 g l_1(1-\cos\varphi_1) + m_2 g(l_1(1-\cos\varphi_1) + l_2(1-\cos\varphi_2)). \tag{3}$$

The generalized forces Q_i effecting the system are expressed as:

$$Q_1 = -m_1 l_1\ddot{x}\cos\varphi_1 - m_2(l_1\cos\varphi_1 + l_2\cos\varphi_2)\ddot{x} + \xi = -((m_1+m_2)l_1\cos\varphi_1 + m_2 l_2\cos\varphi_2)\ddot{x} + \xi, \tag{4}$$

$$Q_2 = -m_2 l_2\ddot{x}\cos\varphi_2\,. \tag{5}$$

If we insert expressions (2)–(5) into equation (1), we get

$$\begin{cases} (m_1+m_2)l_1^2\ddot{\varphi}_1 + m_2l_1l_2\ddot{\varphi}_2\cos(\varphi_1-\varphi_2) + m_2l_1l_2\dot{\varphi}_2^2\sin(\varphi_1-\varphi_2) = \\ = -(m_1+m_2)gl_1\sin\varphi_1 - ((m_1+m_2)l_1\cos\varphi_1 + m_2l_2\cos\varphi_2)\ddot{x} + \xi, \\ m_2l_2^2\ddot{\varphi}_2 + m_2l_1l_2\ddot{\varphi}_1\cos(\varphi_1-\varphi_2) - m_2l_1l_2\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2) = -m_2gl_2\sin\varphi_2 - m_2l_2\ddot{x}\cos\varphi_2. \end{cases} \tag{6}$$

The system (6) may be linearized by assuming, that angles φ_1 and φ_2 are small, i.e. $\cos(\varphi_1-\varphi_2)\approx 1$, $\sin(\varphi_1-\varphi_2)\approx\varphi_1-\varphi_2$, $\sin\varphi_1\approx\varphi_1$, $\sin\varphi_2\approx\varphi_2$. We have

$$\begin{cases} (m_1+m_2)l_1^2\ddot{\varphi}_1 + m_2l_1l_2\ddot{\varphi}_2 = -(m_1+m_2)gl_1\varphi_1 - ((m_1+m_2)l_1 + m_2l_2)\ddot{x} + \xi, \\ m_2l_2^2\ddot{\varphi}_2 + m_2l_1l_2\ddot{\varphi}_1 = -m_2gl_2\varphi_2 - m_2l_2\ddot{x}; \end{cases} \tag{7}$$

$$\begin{cases} \ddot{\varphi}_1 = -\dfrac{m_1+m_2}{m_1}\cdot\dfrac{g}{l_1}\varphi_1 + \dfrac{m_2}{m_1}\cdot\dfrac{g}{l_1}\varphi_2 - \dfrac{m_1l_1+m_2l_2}{m_1l_1^2}\ddot{x} + \dfrac{1}{m_1l_1^2}\xi, \\ \ddot{\varphi}_2 = \dfrac{m_1+m_2}{m_1}\cdot\dfrac{g}{l_2}\varphi_1 - \dfrac{m_1+m_2}{m_1}\cdot\dfrac{g}{l_2}\varphi_2 + \dfrac{m_2}{m_1l_1}\ddot{x} - \dfrac{1}{m_1l_1l_2}\xi. \end{cases} \tag{8}$$

Now make the following notations: $D = -\dfrac{m_1+m_2}{m_1}\cdot\dfrac{g}{l_1} < 0$, $H = \dfrac{m_2}{m_1}\cdot\dfrac{g}{l_1} > 0$, $R = \dfrac{m_1+m_2}{m_1}\cdot\dfrac{g}{l_2} > 0$, $S = -\dfrac{m_1+m_2}{m_1}\cdot\dfrac{g}{l_2} < 0$, $P = -\dfrac{m_1l_1+m_2l_2}{m_1l_1^2}$, $W = \dfrac{m_2}{m_1l_1}$. If we introduce these notations in the set (8), we get

$$\begin{cases} \ddot{\varphi}_1 = D\varphi_1 + H\varphi_2 + P\ddot{x} + \dfrac{1}{m_1l_1^2}\xi, \\ \ddot{\varphi}_2 = R\varphi_1 + S\varphi_2 + W\ddot{x} - \dfrac{1}{m_1l_1l_2}\xi. \end{cases} \tag{9}$$

Let us introduce dimensionless quantities by the following notation: $\tau = kt$, where k has the dimension of frequency. Hence,

$$\begin{cases} \dfrac{d^2\varphi_1}{d\tau^2} = \dfrac{D}{k^2}\varphi_1 + \dfrac{H}{k^2}\varphi_2 + P\dfrac{d^2f(\tau)}{d\tau^2} + \dfrac{1}{k^2m_1l_1^2}\xi, \\ \dfrac{d^2\varphi_2}{d\tau^2} = \dfrac{R}{k^2}\varphi_1 + \dfrac{S}{k^2}\varphi_2 + W\dfrac{d^2f(\tau)}{d\tau^2} - \dfrac{1}{k^2m_1l_1l_2}\xi, \end{cases} \tag{10}$$

where

$$f(\tau) = x\left(\frac{1}{k}\tau\right). \tag{11}$$

Now make the following notations: $d = \dfrac{D}{k^2}$, $h = \dfrac{H}{k^2}$, $r = \dfrac{R}{k^2}$, $s = \dfrac{S}{k^2}$, $u = \dfrac{1}{k^2m_1l_1^2}\xi$, $L = \dfrac{l_1}{l_2}$. Thus, the system (10) can be written as:

$$\begin{cases} \dfrac{d^2\varphi_1}{d\tau^2} = d\varphi_1 + h\varphi_2 + P\dfrac{d^2f(\tau)}{d\tau^2} + u, \\ \dfrac{d^2\varphi_2}{d\tau^2} = r\varphi_1 + s\varphi_2 + W\dfrac{d^2f(\tau)}{d\tau^2} - Lu. \end{cases} \tag{12}$$

If we introduce in system (12) the phase coordinates by means of the following notations: $x_1 = \varphi_1$, $x_2 = \dfrac{d\varphi_1}{d\tau}$, $x_3 = \varphi_2$, $x_4 = \dfrac{d\varphi_2}{d\tau}$, then

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = dx_1 + hx_3 + P\dfrac{d^2f(\tau)}{d\tau^2} + u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = rx_1 + sx_3 + W\dfrac{d^2f(\tau)}{d\tau^2} - Lu: \end{cases} \tag{13}$$

Here for simplicity the notation $\dot{x}_i = \dfrac{dx_i}{d\tau}$ (i=1,2,3,4) was adopted. The obtained system (13) is a controlled system of linear nonhomogeneous differential equations.

For the existence and uniqueness of the solution of system (13) it is necessary and sufficient that the system be completely controlled, i.e. $\operatorname{rank} K = 4$. K is the Kalman matrix of the form

$$K = \begin{pmatrix} 0 & 1 & 0 & d - Lh \\ 1 & 0 & d - Lh & 0 \\ 0 & -L & 0 & r - Ls \\ -L & 0 & r - Ls & 0 \end{pmatrix}.$$

It is obvious, that $\operatorname{rank} K = 4$. Hence, the solution u^0 of system (13) exists and is unique [9].

Now, let us formulate the optimum stabilization problem for the system (13).

Problem. It is required to find an optimal control action u^0 such that it makes the solution $x_1 = x_2 = x_3 = x_4 = 0$ of system (13) asymptotically stable and satisfies the constraint

$$\mathcal{J}[u] = \int_0^\infty (x_2^2 + x_4^2 + u^2) d\tau \to \min . \tag{14}$$

Solution. For the solution of the problem an assumption is made that the double pendulum is composed of two pendulums of equal masses m with equal lengths l, and the suspension point is moving, according to $x = ne^{-\alpha t}$ $(n \neq 0)$ law in the horizontal direction.

In this case, the equations of motion (13) become

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 + x_3 - 2pe^{-\alpha\tau/k} + u, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = 2x_1 - 2x_3 + pe^{-\alpha\tau/k} - u. \end{cases} \tag{15}$$

Here we have $k = \sqrt{\dfrac{g}{l}}$, $p = \dfrac{n\alpha^2}{gk^2} = \dfrac{nl\alpha^2}{g^2}$, $u = \dfrac{1}{ml^2k^2}\xi$, and for simplicity we make the following notation $\dot{x}_i = \dfrac{dx_i}{d\tau}$ (i=1,2,3,4). To solve the optimal stabilization problem we use the Lyapunov–Bellman method [9]. In this case the Bellman expression can be written as

$$B[\cdot] = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_3}\dot{x}_3 + \frac{\partial V}{\partial x_4}\dot{x}_4 + x_2^2 + x_4^2 + u^2 , \tag{16}$$

where $V = V(x_1, x_2, x_3, x_4, \tau)$ is the Lyapunov function for system (15). Now, substitute expressions for $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and $\dot{x}_4$ from (15) into (16). We get

$$B[\cdot]=\frac{\partial V}{\partial \tau}+\frac{\partial V}{\partial x_1}x_2+\frac{\partial V}{\partial x_2}\left(-2x_1+x_3-2pe^{-\frac{\alpha}{k}\tau}+u\right)+\frac{\partial V}{\partial x_3}x_4+ \\ +\frac{\partial V}{\partial x_4}\left(2x_1-2x_3+pe^{-\frac{\alpha}{k}\tau}-u\right)+x_2^2+x_4^2+u^2. \quad (17)$$

As $\left.\frac{\partial B[\cdot]}{\partial u}\right|_{u=u^0}=0$, we have $\left.\frac{\partial B[\cdot]}{\partial u}\right|_{u=u^0}=\frac{\partial V}{\partial x_2}-\frac{\partial V}{\partial x_4}+2u^0=0$. Thus, we obtain

$$u^0=-\frac{1}{2}\left(\frac{\partial V}{\partial x_2}-\frac{\partial V}{\partial x_4}\right). \quad (18)$$

As in the case of optimal control $u=u^0$, the Bellman expression assumes the value of 0, we can substitute (18) in (17) and require it to be equal 0. Hence, we obtain

$$\left.B[\cdot]\right|_{u=u^0}=\frac{\partial V}{\partial \tau}+\frac{\partial V}{\partial x_1}x_2+\frac{\partial V}{\partial x_2}\left(-2x_1+x_3-2pe^{-\frac{\alpha}{k}\tau}\right)+\frac{\partial V}{\partial x_3}x_4+ \\ +\frac{\partial V}{\partial x_4}\left(2x_1-2x_3+pe^{-\frac{\alpha}{k}\tau}\right)+x_2^2+x_4^2-\frac{1}{4}\left(\frac{\partial V}{\partial x_2}-\frac{\partial V}{\partial x_4}\right)^2=0. \quad (19)$$

The Lyapunov function is sought for in the following form [9]:

$$V(x_1,x_2,x_3,x_4,\tau)=V^{(2)}(x_1,x_2,x_3,x_4)+V^{(1)}(x_1,x_2,x_3,x_4,\tau)+V^{(0)}(\tau), \quad (20)$$

where the function $V^{(2)}(x_1,x_2,x_3,x_4)$ is a quadratic form depending on variables x_1, x_2, x_3, x_4, the $V^{(1)}(x_1,x_2,x_3,x_4,\tau)$ is a linear function of variables x_1, x_2, x_3, x_4 with variable coefficients, and $V^{(0)}(\tau)$ is a continuous function of only time τ.

Now introduce the expression of Lyapunov function (20) into (19). We have

$$\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial \tau}+\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial x_1}x_2+\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial x_2}\left(-2x_1+x_3-2pe^{-\frac{\alpha}{k}\tau}\right)+ \\ +\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial x_3}x_4+\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial x_4}\left(2x_1-2x_3+pe^{-\frac{\alpha}{k}\tau}\right)+x_2^2+x_4^2- \quad (21) \\ -\frac{1}{4}\left(\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial x_2}-\frac{\partial(V^{(2)}+V^{(1)}+V^{(0)})}{\partial x_4}\right)^2=0.$$

Now separate the second order terms depending on variables x_1, x_2, x_3, x_4, the linear terms depending on variables x_1, x_2, x_3, x_4 and the terms independent of variables x_1, x_2, x_3, x_4 in equation (21) and equate them to 0 separately. For the second order terms depending on variables x_1, x_2, x_3, x_4, we have

$$\frac{\partial V^{(2)}}{\partial x_1}x_2+\frac{\partial V^{(2)}}{\partial x_2}(-2x_1+x_3)+\frac{\partial V^{(2)}}{\partial x_3}x_4+\frac{\partial V^{(2)}}{\partial x_4}(2x_1-2x_3)+x_2^2+x_4^2-\frac{1}{4}\left(\frac{\partial V^{(2)}}{\partial x_2}-\frac{\partial V^{(2)}}{\partial x_4}\right)^2=0 \quad (22)$$

for linear terms depending on variables x_1, x_2, x_3, x_4, we have

$$\frac{\partial V^{(1)}}{\partial \tau}+\frac{\partial V^{(1)}}{\partial x_1}x_2-2pe^{-\frac{\alpha}{k}\tau}\frac{\partial V^{(2)}}{\partial x_2}+\frac{\partial V^{(1)}}{\partial x_2}(-2x_1+x_3)+\frac{\partial V^{(1)}}{\partial x_3}x_4+pe^{-\frac{\alpha}{k}\tau}\frac{\partial V^{(2)}}{\partial x_4}+ \\ +\frac{\partial V^{(1)}}{\partial x_4}(2x_1-2x_3)-\frac{1}{2}\left(\frac{\partial V^{(2)}}{\partial x_2}-\frac{\partial V^{(2)}}{\partial x_4}\right)\left(\frac{\partial V^{(1)}}{\partial x_2}-\frac{\partial V^{(1)}}{\partial x_4}\right)=0. \qquad (23)$$

And for terms depending only on time τ, we have

$$\frac{dV^{(0)}}{d\tau}-2pe^{-\frac{\alpha}{k}\tau}\frac{\partial V^{(1)}}{\partial x_2}+pe^{-\frac{\alpha}{k}\tau}\frac{\partial V^{(1)}}{\partial x_4}-\frac{1}{4}\left(\frac{\partial V^{(1)}}{\partial x_2}-\frac{\partial V^{(1)}}{\partial x_4}\right)^2=0. \qquad (24)$$

By solving the obtained system of equations (22)–(24), we find $V^{(2)}(x_1,x_2,x_3,x_4)$, $V^{(1)}(x_1,x_2,x_3,x_4,\tau)$ and $V^{(0)}(\tau)$ functions.

As, in the case of $f(\tau)=0$, the system (13) is completely controlled, the function $V^{(2)}(x_1,x_2,x_3,x_4)$ can be uniquely determined. We seek the function in the following form:

$$V^{(2)}(x_1,x_2,x_3,x_4)=\frac{1}{2}(c_{11}x_1^2+c_{22}x_2^2+c_{33}x_3^2+c_{44}x_4^2+2c_{12}x_1x_2+2c_{13}x_1x_3+ \\ +2c_{14}x_1x_4+2c_{23}x_2x_3+2c_{24}x_2x_4+2c_{34}x_3x_4). \qquad (25)$$

For determination of coefficients c_{ij} (i,j=1,2,3,4), we have the following system of algebraic equations:

$$\begin{cases} -2c_{12}+2c_{14}-\frac{1}{4}(c_{12}-c_{14})^2=0, \\ 1+c_{12}-\frac{1}{4}(c_{22}-c_{24})^2=0, \\ c_{23}-2c_{34}-\frac{1}{4}(c_{23}-c_{34})^2=0, \\ 1+c_{34}-\frac{1}{4}(c_{24}-c_{44})^2=0, \\ c_{11}-2c_{22}+2c_{24}-\frac{1}{2}(c_{12}-c_{14})(c_{22}-c_{24})=0, \\ c_{12}-2c_{14}-2c_{23}+2c_{34}-\frac{1}{2}(c_{12}-c_{14})(c_{23}-c_{34})=0, \\ c_{13}-2c_{24}+2c_{44}-\frac{1}{2}(c_{12}-c_{14})(c_{24}-c_{44})=0, \\ c_{13}+c_{22}-2c_{24}-\frac{1}{2}(c_{22}-c_{24})(c_{23}-c_{34})=0, \\ c_{14}+c_{23}-\frac{1}{2}(c_{22}-c_{24})(c_{24}-c_{44})=0, \\ c_{24}+c_{33}-2c_{44}-\frac{1}{2}(c_{23}-c_{34})(c_{24}-c_{44})=0. \end{cases} \qquad (26)$$

From solutions of system (26) we choose that one, in case of which, the function $V^{(2)}(x_1,x_2,x_3,x_4)$ is positively defined. For our system the solution has the following form:

$$c_{11}^0 \approx 6.515172,\ c_{12}^0 \approx 1.652967,\ c_{13}^0 \approx -0.209687,\ c_{14}^0 \approx 1.652967,$$
$$c_{22}^0 \approx 7.651655,\ c_{23}^0 \approx -1.823735,\ c_{24}^0 \approx 4.394069,\ c_{33}^0 \approx 4.647082, \quad (27)$$
$$c_{34}^0 \approx -0.997252,\ c_{44}^0 \approx 4.498912$$

(these coefficients were determined by means of computer program Mathematica 7 to within 10^{-6} accuracy).

Now insert these coefficients in expression (25). We obtain

$$V^{(2)}(x_1,x_2,x_3,x_4) = \frac{1}{2}(c_{11}^0x_1^2 + c_{22}^0x_2^2 + c_{33}^0x_3^2 + c_{44}^0x_4^2 + 2c_{12}^0x_1x_2 + 2c_{13}^0x_1x_3 +$$
$$+2c_{14}^0x_1x_4 + 2c_{23}^0x_2x_3 + 2c_{24}^0x_2x_4 + 2c_{34}^0x_3x_4) = 3.257586x_1^2 + 3.825827x_2^2 + \quad (28)$$
$$+2.323541x_3^2 + 2.249456x_4^2 + +1.652967x_1x_2 - 0.209687x_1x_3 + 1.652967x_1x_4 -$$
$$-1.823735x_2x_3 + 4.394069x_2x_4 - 0.997252x_3x_4.$$

Now determine $V^{(1)}(x_1,x_2,x_3,x_4,\tau)$ function from equation (23). If (28) is introduced in (23), we have

$$\frac{\partial V^{(1)}}{\partial \tau} + \frac{\partial V^{(1)}}{\partial x_1}x_2 + \frac{\partial V^{(1)}}{\partial x_2}(-2x_1 + x_3) + \frac{\partial V^{(1)}}{\partial x_3}x_4 + \frac{\partial V^{(1)}}{\partial x_4}(2x_1 - 2x_3) +$$
$$+pe^{-\frac{\alpha}{k}\tau}(-2(c_{22}^0x_2 + c_{12}^0x_1 + c_{23}^0x_3 + c_{24}^0x_4) + (c_{44}^0x_4 + c_{14}^0x_1 + c_{24}^0x_2 + c_{34}^0x_3)) - \quad (29)$$
$$-\frac{1}{2}((c_{22}^0x_2 + c_{12}^0x_1 + c_{23}^0x_3 + c_{24}^0x_4) - (c_{44}^0x_4 + c_{14}^0x_1 + c_{24}^0x_2 + c_{34}^0x_3))\left(\frac{\partial V^{(1)}}{\partial x_2} - \frac{\partial V^{(1)}}{\partial x_4}\right) = 0.$$

The function $V^{(1)}(x_1,x_2,x_3,x_4,\tau)$ is sought in the following form:

$$V^{(1)}(x_1,x_2,x_3,x_4,\tau) = y_1(\tau)x_1 + y_2(\tau)x_2 + y_3(\tau)x_3 + y_4(\tau)x_4, \quad (30)$$

where $y_1(\tau)$, $y_2(\tau)$, $y_3(\tau)$ and $y_4(\tau)$ are unknown functions of τ. Inserting (30) in (29), we have

$$\dot{y}_1x_1 + \dot{y}_2x_2 + \dot{y}_3x_3 + \dot{y}_4x_4 + y_1x_2 + y_2(-2x_1 + x_3) + y_3x_4 + y_4(2x_1 - 2x_3) +$$
$$+pe^{-\frac{\alpha}{k}\tau}((-2c_{12}^0 + c_{14}^0)x_1 + (-2c_{22}^0 + c_{24}^0)x_2 + (-2c_{23}^0 + c_{34}^0)x_3 + (-2c_{24}^0 + c_{44}^0)x_4) - \quad (31)$$
$$-\frac{1}{2}(y_2 - y_4)((c_{12}^0 - c_{14}^0)x_1 + (c_{22}^0 - c_{24}^0)x_2 + (c_{23}^0 - c_{34}^0)x_3 + (c_{24}^0 - c_{44}^0)x_4) = 0.$$

Now separate the coefficients of variables x_1, x_2, x_3, x_4 and equate them to 0 separately. Thus, we obtain a system of linear nonhomogeneous differential equations for determination of $y_1(\tau)$, $y_2(\tau)$, $y_3(\tau)$ and $y_4(\tau)$ functions:

$$\begin{cases} \dot{y}_1 = a_1y_2 - a_1y_4 - pb_1e^{-\frac{\alpha}{k}\tau}, \\ \dot{y}_2 = -y_1 + a_2y_2 - a_2y_4 - pb_2e^{-\frac{\alpha}{k}\tau}, \\ \dot{y}_3 = a_3y_2 - (a_3 - 1)y_4 - pb_3e^{-\frac{\alpha}{k}\tau}, \\ \dot{y}_4 = a_4y_2 - y_3 - a_4y_4 - pb_4e^{-\frac{\alpha}{k}\tau}, \end{cases} \quad (32)$$

where the following notations were made:

$$2+\frac{c_{12}^0-c_{14}^0}{2}=a_1,\quad \frac{c_{22}^0-c_{24}^0}{2}=a_2,\quad -1+\frac{c_{23}^0-c_{34}^0}{2}=a_3,\quad \frac{c_{24}^0-c_{44}^0}{2}=a_4,$$

$$-2c_{12}^0+c_{14}^0=b_1,\quad -2c_{22}^0+c_{24}^0=b_2,\quad -2c_{23}^0+c_{34}^0=b_3,\quad -2c_{24}^0+c_{44}^0=b_4.$$

We have for numerical values of coefficients

$$a_1\approx 2,\ a_2\approx 1.628793,\ a_3\approx -1.413242,\ a_4\approx -0.052422,$$

$$b_1\approx -1.65297,\ b_2\approx -10.9092,\ b_3\approx 2.65022,\ b_4\approx -4.28923. \qquad (33)$$

Here for simplicity we adopted the following notations $\dot{x}_i=\frac{dx_i}{d\tau}$ (i=1,2,3,4). The solution of system (32) is determined by using the Cauchy formula [10]:

$$y_1^o(\tau)=kpe^{-\frac{\alpha}{k}\tau}\left(\frac{-1.900079+3.951708k}{1+0.228166k+0.569432k^2}+\frac{0.247112+9.213856k}{1+1.45304k+3.51227k^2}\right), \qquad (34)$$

$$y_2^o(\tau)=kpe^{-\frac{\alpha}{k}\tau}\left(\frac{-8.487157-0.265233k}{1+0.228166k+0.569432k^2}+\frac{-2.422084+3.939016k}{1+1.45304k+3.51227k^2}\right), \qquad (35)$$

$$y_3^o(\tau)=kpe^{-\frac{\alpha}{k}\tau}\left(\frac{1.826889+3.612668k}{1+0.228166k+0.569432k^2}+\frac{0.82333-7.065952k}{1+1.45304k+3.51227k^2}\right), \qquad (36)$$

$$y_4^o(\tau)=kpe^{-\frac{\alpha}{k}\tau}\left(\frac{-6.294537+0.275749k}{1+0.228166k+0.569432k^2}+\frac{-2.005312+0.591238k}{1+1.45304k+3.51227k^2}\right). \qquad (37)$$

The optimal function $V^{(1)}(x_1,x_2,x_3,x_4,\tau)$ is determined by inserting $y_1(\tau)$, $y_2(\tau)$, $y_3(\tau)$ and $y_4(\tau)$ functions from (34), (35), (36), (37) in (30).

As for the optimal control action we have expression (18) and the function $V^{(0)}$ is only τ dependent, i.e. $\frac{\partial V^{(0)}}{\partial x_2}=\frac{\partial V^{(0)}}{\partial x_4}\equiv 0$, then we may omit the calculation of $V^{(0)}$ function. Consequently, the optimal control action can be determined by means of the following expression:

$$u^0=-\frac{1}{2}\left(\frac{\partial V^{(2)}}{\partial x_2}-\frac{\partial V^{(2)}}{\partial x_4}+\frac{\partial V^{(1)}}{\partial x_2}-\frac{\partial V^{(1)}}{\partial x_4}\right). \qquad (38)$$

Now insert (25) and (30) into (38). We obtain

$$u^0=-\frac{1}{2}((c_{12}^0-c_{14}^0)x_1+(c_{22}^0-c_{24}^0)x_2+(c_{23}^0-c_{34}^0)x_3+(c_{24}^0-c_{44}^0)x_4)- -\frac{1}{2}y_2^0(\tau)+\frac{1}{2}y_4^0(\tau). \qquad (39)$$

If we insert (27), (35) and (37) in (40) and do simplifications, then we obtain for u^0

$$u^0(x_1,x_2,x_3,x_4,\tau)=-1.62879x_2+0.413242x_3+0.0524218x_4- -kpe^{-\frac{\alpha}{k}\tau}\frac{(-1.990251+0.001563k)(0.327772+0.059833k+k^2)}{(1.756139+0.40069k+k^2)(0.284716+0.413706k+k^2)}. \qquad (41)$$

Thus, the optimal control action u^0 determined by means of expression (41) is the required control action.

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