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Mathematics

ON CONVERGENCE IN $L_1[0,1]$ NORM OF SOME IRREGULAR LINEAR MEANS OF WALSH–FOURIER SERIES

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In this paper the convergence in $L_1[0,1]$ of some irregular linear means of Fourier–Walsh series of integrable functions after correcting these functions on sets of small measure is studied.

***Keywords*:** triangular matrix, irregular linear means, Dirichlet kernels.

Introduction. First recall the definition of linear triangular methods of summation for arbitrary numerical series. Consider the following numerical series

$$\sum_{k=0}^{\infty} u_k. \tag{1}$$

By S_k, $k=0,1,\ldots$, we denote the partial sums of this series. Let $T=\|a_{mk}\|$ be any infinite triangular matrix, i.e. matrix satisfying $a_{mk}=0,\ k>m,\ m=0,1,\ldots$ The series (1) is said to be summable by the method defined by matrix *T*, or shorter, *T*-summable to the value *S*, if

$$\lim_{m\to\infty} T_m = S, \qquad T_m = \sum_{k=0}^{m} a_{mk} S_k. \tag{2}$$

T_m is called the *T*-mean of the series (1). Summation method is called *regular*, if every convergent series is summable by this method to its sum. The following theorem is well known:

Theorem (Teoplitz). The conditions

1) $\lim\limits_{m\to\infty} a_{mk}=0$ for any fixed k;

2) $\lim\limits_{m\to\infty} \sum\limits_{k=0}^{m} a_{mk}=1$;

3) $\exists H>0$ s.t. $\sum\limits_{k=0}^{m} |a_{mk}|<H$ for all m

are necessary and sufficient for the regularity of the *T*-method.

In [2] D.E. Menshov introduced the following class of irregular in general summation methods.

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Definition. Let $\beta > 0$. Triangular method of summation T is called of R^{β}-type, if the elements of the matrix T satisfy the conditions 1), 2) of the previous Theorem and $\exists M > 0$ such that $|a_{mm}| < Mm^{\beta}$, $|a_{mk}| < \frac{Mm^{\beta}}{(m-k)^{\beta+1}}$, $0 \le k < m$. For the trigonometric system Menshov proved the following:

Theorem (D.E. Menshov). Let T be a triangular method of summation of the type R^{β}. For any integrable function $f(x)$ and for any perfect nowhere dense set $P \subset [-\pi, \pi]$ there exists an integrable function $g(x)$ and a sequence of natural numbers m_j such that

$$1)\ f(x) = g(x), \quad x \in P; \qquad 2)\ \lim_{j\to\infty} T_{m_j}(x, g) \overset{a.e.}{=} g(x).$$

Now we will give the definition of the Walsh system (see [1]). The Walsh system $\{w_k\}_{k=0}^{\infty}$ consists of the following functions:

$$w_0(x) = 1, \quad w_n(x) = \prod_{s=1}^{k} r_{m_s}(x), \quad n = \sum_{s=1}^{k} 2^{m_s}, \quad m_1 > m_2 > \ldots > m_s,$$

where $\{r_k(x)\}_{k=0}^{\infty}$ is the Rademacher system, defined by

$$r_0(x) = \begin{cases} 1, & x \in [0, 1/2), \\ -1, & x \in [1/2, 1), \end{cases} \quad r_0(x) = r_0(x+1), \quad r_k(x) = r_0(2^k x), \quad k = 1, 2, \ldots$$

We will call the T-method to be of R-type, if it satisfies conditions 1), 2) of the Teoplitz's Theorem. In this paper we prove the following:

Theorem. Let T be a triangular method of summation of the type R. Let $\{M_j\}_{j=1}^{\infty}$ and $\{\omega_j\}_{j=1}^{\infty}$ be given increasing sequences of naturals. Then for any $\varepsilon > 0$ there exists a set E with measure $|E| > 1 - \varepsilon$ such that for any integrable function $f(x)$ there exist an integrable function $g(x)$ coinciding with $f(x)$ on E and a sequence of natural numbers $\{q_p\}_{p=1}^{\infty}$ such that

$$\lim_{m\in\Omega;\, m\to\infty} \int_0^1 |\tilde{\sigma}_m(x, g) - g(x)| dx = 0, \quad \text{where} \quad \Omega = \bigcup_{\nu=1}^{\infty} [M_{q_\nu} - \omega_\nu, M_{q_\nu}].$$

Auxiliary Results. We use the constructions introduced by M.G. Grigorian in [3, 4] to prove the following lemmas.

Lemma 1. Let numbers $N_0 > 1$, $\gamma \ne 0, \nu_0$, dyadic interval $\Delta = \Delta_j^{(p)} = \left[\frac{j-1}{2^p}, \frac{j}{2^p}\right)$ and a triangular matrix $T = \| a_{mk} \|$ are given. Then there exist a set $E \subset \Delta$ and a polynomial in the Walsh system of the form

$$Q(x) = \sum_{k=N_0}^{N} c_k w_k(x) \tag{3}$$

such that

$$1)\ |E| = |\Delta| (1 - 2^{-\nu_0}), \qquad 2)\ Q(x) = \begin{cases} \gamma, & x \in E, \\ 0, & x \notin \Delta; \end{cases}$$

3) $\max\limits_{N_0\le q\le N}\left\|\sum\limits_{k=N_0}^{q} c_k w_k(x)\right\|_{L_1} \le 2|\gamma|\sqrt{2^{\nu_0}|\Delta|}$, 4) $\|Q(x)\|_{L_1} \le 2|\gamma||\Delta|$;

5) the *T*-means $\tilde{\sigma}_m(x,Q)$ of the Fourier–Walsh series of polynomial $Q(x)$ satisfy the following inequality:

$$\|\tilde{\sigma}_m(x,Q)-Q(x)\|_{L_1} \le \|Q(x)\|_{L_1}\left(2\sum_{k=N_0}^{N-1}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right),\ m>N.$$

Proof. Let $s=[\log_2 N_0]+p.$ Consider the function

$$I_{\nu_0}(x)=\begin{cases}1, & x\in[0,1)\setminus\Delta_1^{(\nu_0)},\\ 1-2^{\nu_0}, & x\in\Delta_1^{(\nu_0)}.\end{cases} \tag{4}$$

Extend this function from [0,1) to the real axis as a periodic function with period 1. We define the function $Q(x)$ in the following manner

$$Q(x)=\gamma I_{\nu_0}(2^s x)\chi_\Delta(x). \tag{5}$$

It is easy to verify that $Q(x)$ is a polynomial in Walsh system, which spectrum lies to the right of 2^s, i.e. $Q(x)$ has the form (3) with $N=\max\{n;\, c_n(Q)\ne 0\}$, where $c_n(Q)$, $n\ge 1$, are Fourier–Walsh coefficients of the polynomial $Q(x)$. Let $E=\{x: Q(x)=\gamma\}$. It is easy to see that $|E|=|\Delta|(1-2^{-\nu_0})$. The validity of 4) follows immediately from (4) and (5). Let us prove the assertion 3). We have

$$\max_{N_0\le q\le N}\left\|\sum_{k=N_0}^{q}c_k w_k(x)\right\|_{L_1} \le \max_{N_0\le q\le N}\left\|\sum_{k=N_0}^{q}c_k w_k(x)\right\|_{L_2} \le \left(\sum_{k=N_0}^{N}c_k^{\,2}\right)^{1/2} = \|Q(x)\|_{L_2} \le 2|\gamma|\sqrt{2^{\nu_0}|\Delta|}.$$

According to the definition, *T*-means of Fourier series of $Q(x)$ for any $m>N$ have the following form

$$\tilde{\sigma}_m(x,Q)=\sum_{k=0}^{m}a_{mk}S_k(x,Q)=\sum_{k=N_0}^{N-1}a_{mk}S_k(x,Q)+Q(x)\sum_{k=N}^{m}a_{mk}, \tag{6}$$

where $S_k(x,Q)$, $k=0,1,\dots,$ are the partial sums of Fourier series of $Q(x)$. Using (6) and the property of convolution operator (see [1], (2.1.6), (2.1.7)), we can write

$$\begin{aligned}\|\tilde{\sigma}_m(x,Q)-Q(x)\|_{L_1} &\le \left\|\sum_{k=N_0}^{N-1}a_{mk}S_k(x,Q)\right\|_{L_1} + \\ +\|Q(x)\|_{L_1}\left|\sum_{k=0}^{N-1}a_{mk}\right| &\le \int_0^1\left|\int_0^1 Q(x\oplus t)K_m(N,N_0,t)dt\right|dx+\|Q(x)\|_{L_1}\left|\sum_{k=0}^{N-1}a_{mk}\right|+ \\ +\|Q(x)\|_{L_1}\left|\sum_{k=0}^{m}a_{mk}-1\right| &\le \|Q(x)\|_{L_1}\left[\|K_m(N,N_0,t)\|_{L_1}+\left|\sum_{k=0}^{m}a_{mk}-1\right|+\left|\sum_{k=0}^{N-1}a_{mk}\right|\right],\end{aligned} \tag{7}$$

where $K_m(N,N_0,t)=\sum\limits_{k=N_0}^{N-1}a_{mk}D_k(t)$, and $D_s(t)$, $s=0,1,\dots,$ are the Dirichlet kernels. Using the estimate for L_1 norms of Dirichlet kernels, we easily obtain

$$\|K_m(N,N_0,t)\|_{L_1} \le \sum_{k=N_0}^{N-1}|a_{mk}|\log_2(k+4).$$

From this inequality and (7) we finally obtain

$$\left\|\tilde{\sigma}_m(x,Q)-Q(x)\right\|_{L_1}\le\left\|Q(x)\right\|_{L_1}\left(2\sum_{k=N_0}^{N-1}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right).$$

This completes the proof of Lemma 1.

Lemma 2. Let numbers $k_0>1$, $\varepsilon\in(0,1)$, Walsh polynomial $f(x)$ (such that $f(x)\ne 0$, $x\in(0,1)$) and triangular matrix $T=\|a_{mk}\|$ are given. Then there exist a set $E\subset[0,1]$ and a polynomial $Q(x)$ of the form $Q(x)=\sum_{k=k_0+1}^{\bar{k}}c_k w_k(x)$ such that

1) $|E|>1-\varepsilon$, 2) $Q(x)=f(x),\quad x\in E$;

3) $\max_{k_0<q<\bar{k}}\left\|\sum_{k=k_0}^{q}c_k w_k(x)\right\|_{L_1}\le 3\left\|f(x)\right\|_{L_1}$, 4) $\left\|Q(x)\right\|_{L_1}\le 2\left\|f(x)\right\|_{L_1}$;

5) $\left\|\tilde{\sigma}_m(x,Q)-Q(x)\right\|_{L_1}\le 2\left\|f(x)\right\|_{L_1}\left(2\sum_{k=0}^{\bar{k}}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right)$,

$m>N$.

Proof. Let $f(x)=\sum_{j=1}^{M}\gamma_j\chi_{\Delta_j}(x)$, where Δ_j is a dyadic interval and $\bigcup_{j=1}^{M}\Delta_j=[0,1)$. Take $\nu_0=1+[\log_2 1/\varepsilon]$. Without loss of generality we can assume

$$\max_{1\le j\le M}|\gamma_j|(2^{\nu_0}|\Delta_j|)^{1/2}<\min\left\{\varepsilon/2;\int_0^1|f(x)|dx/2\right\}. \tag{8}$$

Successively applying Lemma 1, we determine some sets E_j and polynomials $Q_j(x)$,

$$Q_j(x)=\sum_{k=N_{j-1}}^{N_j-1}c_k^{(j)}w_k(x),\qquad j=1,2,...,M,\qquad N_0=k_0+1, \tag{9}$$

which satisfy the following conditions:

$$|E_j|=|\Delta_j|(1-2^{-\nu_0}), \tag{10}$$

$$Q_j(x)=\begin{cases}\gamma_j, & x\in E_j,\\ 0, & x\notin\Delta_j,\end{cases}\qquad j=1,2,...,M, \tag{11}$$

$$\max_{N_{j-1}\le q<N_j}\left\|\sum_{k=N_{j-1}}^{q}c_k w_k(x)\right\|_{L_1}\le 2|\gamma_j|\sqrt{2^{\nu_0}|\Delta_j|},\ \left\|Q_j(x)\right\|_{L_1}\le 2|\gamma_j||\Delta_j|, \tag{12}$$

$$\left\|\tilde{\sigma}_m(x,Q_j)-Q_j(x)\right\|_{L_1}\le\left\|Q_j(x)\right\|_{L_1}\left(2\sum_{k=N_{j-1}}^{N_j-1}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right),\quad m\ge N_j. \tag{13}$$

Let

$$Q(x)=\sum_{j=1}^{M}Q_j(x)=\sum_{j=1}^{M}\sum_{k=N_{j-1}}^{N_j-1}c_k^{(j)}w_k(x)=\sum_{k=k_0+1}^{\bar{k}}c_k w_k(x),\ \bar{k}=N_M-1, \tag{14}$$

and $E=\bigcup_{j=1}^{M}E_j$. Then obviously we get 1) and 2) (see (10), (11)). Let $N_{i-1}\le q\le N_i-1$, then from (8) and (12) we have

$$\left\|\sum_{k=k_0}^{q}c_k w_k(x)\right\|_{L_1}\le\left\|\sum_{j=1}^{i-1}Q_j(x)\right\|_{L_1}+\left\|\sum_{k=N_{i-1}}^{q}c_k^{(i)}w_k(x)\right\|_{L_1}\le 3\|f(x)\|_{L_1}.$$

This proves the validity of 3).

Further, for all $m>\bar{k}$ we have (see (12), (14))

$$\|\tilde{\sigma}_m(x,Q)-Q(x)\|_{L_1}\le\sum_{j=1}^{M}\|\tilde{\sigma}_m(x,Q_j)-Q_j(x)\|_{L_1}\le$$

$$\le\sum_{j=1}^{M}\|Q_j(x)\|_{L_1}\left(2\sum_{k=N_{j-1}}^{N_j-1}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right)\le$$

$$\le 2\|f(x)\|_{L_1}\left(2\sum_{k=0}^{\bar{k}}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right).$$

This completes the proof of Lemma 2.

The Proof of Theorem. Let $\varepsilon>0$, and let $\{f_n(x)\}_{n=1}^{\infty}$ be the sequence of polynomials in the Walsh system with rational coefficients enumerated in some order. Successively applying Lemma 2, we can choose an increasing sequence of positive integers $\{j_\nu\}_{\nu=1}^{\infty}$, the sequence of sets $\{E_n\}_{n=1}^{\infty}$ and polynomials $Q_n(x)=\sum_{s=M_{j_n}}^{\bar{M}_n}c_k w_k(x)$, $M_{j_1}=M_1$, satisfying

$$Q_n(x)=f_n(x),\qquad x\in E_n, \tag{15}$$

$$|E_n|>1-\varepsilon 2^{-n}, \tag{16}$$

$$\|Q_n(x)\|_{L_1}\le 2\,\|f_n(x)\|_{L_1},\qquad \max_{M_{j_n}<q<\bar{M}_n}\left\|\sum_{k=M_{j_n}}^{q}c_k w_k(x)\right\|_{L_1}\le 3\,\|f_n(x)\|_{L_1}, \tag{17}$$

$$\|\tilde{\sigma}_m(x,Q)-Q_n(x)\|_{L_1}\le 2\|f_n(x)\|_{L_1}\left(2\sum_{k=0}^{\bar{M}_n}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right),m>\bar{M}_n,$$

$$M_{j_{k+1}}>2\bar{M}_k,\qquad M_{j_k}>2\omega_{k+1},\quad k=1,2,...,$$
$$\max_{r\in[0,\bar{M}_k]}\{|a_{\nu r}|\log_2(r+4)\}\le\frac{1}{k\bar{M}_k},\qquad \nu\ge\frac{M_{j_{k+1}}}{2},\quad k=1,2,... \tag{18}$$

Let $E=\bigcap_{n=1}^{\infty}E_n$. In the light of (16) we obtain $|E|>1-\varepsilon$. Let $f\in L_1(0,1)$. Choose a subsequence $\{f_{n_p}(x)\}_{p=1}^{\infty}$ such that

$$\lim_{N\to\infty}\left\|\sum_{p=1}^{N}f_{n_p}(x)-f(x)\right\|_{L_1}=0,\quad \lim_{N\to\infty}\sum_{p=1}^{N}f_{n_p}(x)=f(x),\ \text{a.e. on }(0,1)$$

$$\|f_{n_p}(x)\|_{L_1}\le 2^{-p},\ p\ge 2. \tag{19}$$

Let $A=\{x\in[0,1]:\sum_{p=1}^{\infty} f_{n_p}(x)\neq f(x)\}$. Consider the following functions

$$\overline{g}(x)=\sum_{p=1}^{\infty} Q_{n_p}(x)\ ,\ \ g(x)=\begin{cases}\overline{g}(x), & x\notin A,\\ f(x), & x\in A.\end{cases} \tag{20}$$

It follows that $g(x)$ and $\overline{g}(x)$ are integrable. Let $M_{j_{n_p}}=M_{q_p}$, $p=1,2,...$ Finally we consider the series $\sum_{k=1}^{\infty}\delta_k c_k w_k(x)$, where

$$\delta_k=\begin{cases}1, & \text{for } k\in\bigcup_{p=1}^{\infty}(M_{q_p},\bar{M}_{n_p}],\\ 0, & \text{otherwise},\end{cases} \tag{21}$$

now we show that partial sums of the series (21) converge in L_1 norm to the function $g(x)$, implying that the series (21) is the Fourier–Walsh series of $g(x)$. Let $M_{q_p}\le N<M_{q_{p+1}}$. Using (17) and (19) we obtain

$$\left\|\sum_{k=1}^{N}\delta_k c_k w_k(x)-g(x)\right\|_{L_1}\le \max_{M_{q_p}\le i\le \bar{M}_{n_p}}\left\|\sum_{k=M_{q_p}}^{i} c_k w_k(x)\right\|_{L_1}+\sum_{k=p}^{\infty}\left\|Q_{n_p}(x)\right\|_{L_1},$$

whence, by (15)–(18), we have what we need.

Let $\Omega=\bigcup_{p=1}^{\infty}[M_{q_p}-\omega_p,M_{q_p}]$. If $m\in[M_{q_N}-\omega_N,M_{q_N}]$, then, using (17)–(21) and the fact that series (21) is the Fourier–Walsh series of $g(x)$, we have

$$\int_0^1|\tilde{\sigma}_m(x,g)-g(x)|dx\le\sum_{p=1}^{N-1}\left\|\tilde{\sigma}_m(x,Q_{n_p})-Q_{n_p}(x)\right\|_{L_1}+\sum_{k=N}^{\infty}\left\|Q_{n_p}(x)\right\|_{L_1}\le$$

$$\le\sum_{p=1}^{N-1}2\left\|f_{n_p}(x)\right\|_{L_1}\left(2\sum_{k=0}^{\bar{M}_{n_p}}|a_{mk}|\log_2(k+4)+|\sum_{k=0}^{m}a_{mk}-1|\right)+\sum_{k=N}^{\infty}\left\|Q_{n_p}(x)\right\|_{L_1}\to 0.$$

Theorem is proved.

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