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Mathematics

ON ONE SPECTRUM OF UNIVERSALITY FOR WALSH SYSTEM

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In the present work it is shown that the set $D=\left\{\sum_{i=0}^{\infty}\delta_i 2^{N_i}:\delta_i=0,1\right\}$ for every sequence $N_0<N_1<\ldots<N_i<\ldots$ of natural numbers can be changed into the set of the form $\Lambda=\{k+o(\omega(k)):k\in D\}$, where $\omega(k)$ is an arbitrary, tending to infinity at $k\to+\infty$ sequence, such that Λ is the spectrum of universality for Walsh system.

***Keywords*:** Walsh system, universal series, representation theorems, representations by subsystems.

Introduction. Let S be a space of functions defined on $[0,1]$ (for example, $S=L^p[0,1]$) and let T be a type of convergence (for example, the convergence in $L^p[0,1]$ metric or the almost everywhere convergence). Here we will mainly consider $S=L^0[0,1]$ – the class of all almost everywhere finite, measurable functions and $T=$ almost everywhere convergence on $[0,1]$.

A series

$$\sum_{k=1}^{\infty}a_k\varphi_k(x) \tag{1}$$

is said to be *universal in the usual sense* for S, T, if for any function $f(x)\in S$ there exists an increasing sequence of natural numbers n_k, such that the corresponding sequence of partial sums $\sum_{j=1}^{n_k}a_j\varphi_j(x)$ converges to $f(x)$ in the sense of T.

There are also other types of universality such as *universality with respect to rearrangements* for S, T: the latter means that for any function $f(x)\in S$ there exists rearrangement $k\mapsto\sigma(k)$ such that the series $\sum_{k=1}^{\infty}a_{\sigma(k)}\varphi_{\sigma(k)}(x)$ converges to $f(x)$ in the sense of T.

We will also say that the series (1) is universal in the sense of partial series

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for S, T, if for any function $f(x) \in S$ there exists a partial series $\sum_{k=1}^{\infty} a_{n_k} \varphi_{n_k}(x)$ of (1), which converges to $f(x)$ in the sense of T.

The first example of trigonometric series universal in the usual sense for the class of all measurable functions has been constructed by D.E. Menshoff [1] (see also [2]). This result was extended by A.A. Talalian [3] to arbitrary complete orthonormal systems. He also established [4], that if $\{\varphi_n(x)\}_{n=1}^{\infty}$, $x \in [0,1]$, is an arbitrary orthonormal system, then there exist a series $\sum a_k \varphi_k(x)$, which is universal in the sense of partial series for the class of all measurable functions and T =convergence in measure on [0,1]. The following general result was obtained by M. Grigorian [5]:

Theorem. The class of orthogonal series simultaneously possessing the following properties 1), 2) are not empty:

1) universality with respect to rearrangements and in the sense of partial series both in each $L^p[0,1]$, $p \in [1,2)$, and in $\bigcap_{1 \le p < 2} L^p[0,1]$;

2) universality with respect to rearrangements and in the sense of partial series for S =all measurable functions and T =almost everywhere convergence on [0,1].

The fact that there exists a functional series universal with respect to rearrangements for S =class of almost everywhere finite, measurable functions and T =almost everywhere convergence, was mentioned by W. Orlicz [6]. Note that Riemann has proved (see [7], p. 317) that every unconditionally convergent numerical series is universal with respect to rearrangements for S = all reals.

Definition. The set of natural numbers Λ, for which it is possible to construct an universal (in some sense) series $\sum_{\lambda_k \in \Lambda} a_k \varphi_{\lambda_k}(x)$, we will call the spectrum of universality (in the same sense).

In the rest of the paper we will consider universal series in Walsh system.

Let $\omega(k)$ be an arbitrary sequence, tending to infinity as $k \to +\infty$. By the small change of some set D we will mean the set $\{k + o(\omega(k)) : k \in D\}$.

Such small transformations of sets were considered for the first time by G. Kozma and A. Olevskii [8], with the aim to transform these sets into representation spectrum. More precisely, it was proved by them for trigonometric system that for any sequence $w(k)$ tending to infinity there is a symmetric representation spectrum $\Lambda = \left\{\pm k^2 + o(w(k))\right\}_{k \in N}$, i.e. each measurable function f allows the representation $f(x) = \sum_{n \in \Lambda} c_n(f) e^{inx}$, where the sum converges almost everywhere.

This result was extended to the Walsh system by the author in [9], namely:

Theorem. For arbitrary $l \in \{2^k\}_{k=0}^{\infty}$ there exists a subsystem $\{w_{n_k}\}_{k=1}^{\infty}$, $n_k \in \{k^l + o(k^{l-1})\}_{k \in N}$ of Walsh system such that for every measurable function there exists a series by subsystem $\{w_{n_k}\}_{k=1}^{\infty}$ converging a.e. to this function. In other words, there exists a representation spectrum of the form $\Lambda_l = \left\{k^l + o(k^{l-1})\right\}_{k \in N}$, $l \in \{2^k\}_{k=0}^{\infty}$.

Theorem. For arbitrary sequence $\{\omega(k)\}_{k=1}^{\infty}$, tending to infinity, there exists a subsystem $\{w_{n_k}\}_{k=1}^{\infty}$, $n_k \in \{k^2 + o(\omega(k))\}_{k\in N}$, of Walsh system such that for arbitrary measurable function there exists a series by subsystem $\{w_{n_k}\}_{k=1}^{\infty}$ converging a.e. to this function, i.e. there exists a representation spectrum $\Lambda = \left\{k^2 + o(\omega(k))\right\}_{k\in N}$ (the notation $\{k^2 + o(\omega(k))\}_{k\in N}$ means that we can find a sequence $\alpha_k \to 0$ such that $\{k^2 + \alpha_k \cdot \omega(k))\}_{k\in N}$ is a representations spectrum).

Let us consider the set of natural numbers in binary representation: $N = \left\{ \sum\limits_{i=0}^{\infty} \delta_i 2^i : \delta_i = 0,1 \right\}$. After substituting all indexes i in the exponents by N_i (for a given sequence $N_0 < N_1 < \ldots N_i < \ldots$) we will get the set $D = \left\{ \sum\limits_{i=0}^{\infty} \delta_i 2^{N_i} : \delta_i = 0,1 \right\}$, which, as it can be easily seen, cannot be a universality spectrum in general. However, for any sequence $\omega(k)$, tending to infinity, by small change of D it can be transformed into a spectrum of universality for the Walsh system. The main result of the present work is the following

Theorem. For any sequence of nonnegative integers $N_0 < N_1 < \ldots < N_i < \ldots$ and arbitrary sequence $\omega(k)$, tending to infinity, the set $D = \left\{ \sum\limits_{i=0}^{\infty} \delta_i 2^{N_i} : \delta_i = 0,1 \right\}$ can be transformed into the set $\Lambda = \{k + o(\omega(k)) : k \in D\} = \{\lambda_n\}_{n=1}^{\infty}$ by small change such that Λ is a universality spectrum (in $S = L^0[0,1]$ and in the sense of T =convergence almost everywhere) for Walsh system, i.e. there exists a series $\sum\limits_{k=1}^{\infty} a_k\, w_{\lambda_k}(x)$ with $a_i \to 0$, such that for arbitrary function $f \in L^0[0,1]$ there is a sequence of natural numbers $\{\nu_k\}$ such that $\lim\limits_{k\to\infty} \sum\limits_{i=1}^{\nu_k} a_i w_{\lambda_i}(x) = f(x)$ almost everywhere on $[0,1]$.

Definitions, Notations and Some Properties. Let us recall the definition of Walsh system $\{w_k(t)\}_{k=0}^{\infty}$ in the Paley ordering [10, 11]:

$$w_0(t) = 1, \quad w_1(t) = \begin{cases} 1, & t \in [0,\ 1/2), \\ -1, & t \in [1/2,\ 1], \end{cases} \qquad w_{2^k}(t) = w_1(2^k t),$$

and for natural q with binary representation $q = \sum\limits_{i=0}^{\infty} q_i 2^i$, where $q_i = 0$ or $q_i = 1$, we define $w_q(t) = \prod\limits_{i=0}^{\infty} \left(w_{2^i}(t)\right)^{q_i}$. Using this definition, it is easy to check the following properties, which we will use later in the text:

1) for every natural number q we have $w_q(2^k t) = w_{q\cdot 2^k}(t)$;

2) if natural numbers p and q have nonintersecting binary representation

(see definition below), then $w_p(t)w_q(t)=w_{p+q}(t)$ (the property of index addition).

Let $p=2^{i_0}+\dots+2^{i_k}$ and $q=2^{j_0}+\dots+2^{j_n}$ be some natural numbers. We will say that binary representations of numbers p and q do not intersect, if $\{i_0,\dots,i_k\}\cap\{j_0,\dots,j_n\}=\varnothing$.

Let $f(t)\in L[0,1]$ and $\hat f(k)=\int_0^1 f(t)\,w_k(t)\,dt$ be its Fourier–Walsh coefficient. Then for each polynomial $P(t)$ in Walsh system we have:

$$P(t)=\sum_{k\ge 0}\hat P(k)\,w_k(t); \qquad (*)$$

1. $(P)_m=\sum_{k=0}^{m}\hat P(k)\,w_k$;
2. $\operatorname{spec}\{P\}$ represents the set of those nonnegative integers k, for which w_k appears in the representation (*);
3. $\deg\{P\}$ is the maximal element of $\operatorname{spec}\{P\}$;
4. $\|\hat P\|_1=\sum_{k\in\operatorname{spec}\{P\}}|\hat P(k)|$.

The Construction of the Spectrum of Universality. For the given sequence $N=\{N_0,N_1,\dots,N_k,\dots\}$ of increasing nonnegative integers we define the following sets:

$$S(i,n)=\left\{\sum_{k=0}^{n}\delta_k 2^{N_k^{(i,n)}}:\delta_k=0,1;\ N_k^{(i,n)}\in N\right\} \text{ and } B_n=\bigcup_{i=0}^{n}\big(i+S(i,n)\big),$$

where $N_k^{(i,n)}$ are chosen such that the following conditions are satisfied:

1. $\dfrac{n}{\omega(\min\{S(i,n)\})}<\dfrac{1}{n}$ for all $0\le i\le n$;
2. $\max\{S(i-1,n)\}<\min\{S(i,n)\},\ 1\le i\le n;$
3. $\max\{B_{n-1}\}<\min\{B_n\}$.

Then, for sufficiently large n, we have $B_n=\{k+o(\omega(k)):k\in D\}$. Note that $S=\bigcup_{n=0}^{\infty}\bigcup_{i=0}^{n}\big(S(i,n)\big)\subset D$ and small change of it a subset D is specified. Other elements of D will be changed by 0, which is also a special case of small change. Thus, $\Lambda'=\bigcup_{n=0}^{\infty}B_n=\{k_m+o(\omega(k_m)):k_m\in D\}=\{\lambda_n\}_{n=1}^{\infty}\subset\{k+o(\omega(k)):k\in D\}=\Lambda.$

We will prove that Λ' is a universality spectrum, which means that Λ is a universality spectrum too. To prove that Λ' is a universality spectrum, it is enough to prove the following lemma.

Main Lemma. For every $f\in L^0[0,1]$ and for arbitrary $\varepsilon>0,\ \delta>0$ and $k_0\in N$ there exists a polynomial $P(x)$ in Walsh system such that:

1. $P(x)=\sum_{k=k_0}^{\bar k}a_k w_{\lambda_k}(x);$
2. $\lambda_k\in\Lambda;$

3. $|a_k|<\delta$;
4. $\operatorname{mes}\{|f(x)-P(x)|>\delta\}<\varepsilon$.

Proof of the Main Lemma. First we need to prove the following lemma.

Lemma. For any $|a|<1,\ 0<\alpha<1,\ y>0$ and any $N_i\in\mathbb{N}$ with $N_0<N_1<\ldots<N_{k-1}$ there exists a polynomial $W(t)=\sum_{i=1}^{2^k-1}\hat{W}(i)\overline{w}_i(t)$ such that:

1. $m\{t:|1-W(t)|\geq y\}<y^{-\alpha}c^k$;
2. $|\hat{W}(i)|\leq a$,

where $c=\dfrac{(1-a)^\alpha+(1+a)^\alpha}{2}<1$ and $\overline{w}_i(t)=w_{q_0 2^{N_0}+\cdots+q_{k-1}2^{N_{k-1}}}(t)$ for $i=q_0 2^0+\cdots+q_{k-1}2^{k-1},\quad q_j=0,1,\quad 0\leq j\leq k-1$.

In the rest of the paper to emphasize that the polynomial $W(t)$ in the Lemma depends on numbers $N_0,N_1,\ldots,N_{k-1}$, we will denote $W(t)=W(t)\{N_0,\ldots,N_{k-1}\}$.

Proof. For the natural numbers $N_0<N_1<\ldots<N_{k-1}$ we denote $\varphi_m(t)=a\cdot w_1\left(2^{N_{m-1}}t\right)=a\cdot w_{2^{N_{m-1}}}(t)$ with $|a|<1$, then $\varphi_k=a$ on the first half of each interval $\Delta_i^{(k)}=\left[\dfrac{i-1}{2^{N_{k-1}}},\dfrac{i}{2^{N_{k-1}}}\right]$, $1\leq i\leq 2^{N_{k-1}}$, and $\varphi_k=-a$ on the second half. Now for $\alpha<1$ we have

$$\int_{\Delta_i^{(k)}}(1-\varphi_k(t))^\alpha dt=\frac{|\Delta_i^{(k)}|}{2}((1-a)^\alpha+(1+a)^\alpha)=c\int_{\Delta_i^{(k)}}dt,$$

where we denote $c=\dfrac{(1-a)^\alpha+(1+a)^\alpha}{2}<\left(\dfrac{1-a+1+a}{2}\right)^\alpha=1$.

It is easy to see that φ_j, for $0\leq j<k-1$, are constant on each of $\Delta_i^{(k)}$, $1\leq i\leq 2^{N_{k-1}}$. Let us prove that $\int_0^1(1-\varphi_1(t))^\alpha\ldots(1-\varphi_n(t))^\alpha dt=c^n$.

For $n=1$ it is obvious. Let us assume that the statement is true for $n=k-2$ and prove it for $n=k-1$. We have

$$\int_0^1(1-\varphi_1(t))^\alpha\ldots(1-\varphi_{k-1}(t))^\alpha dt=\sum_{i=1}^{2^{N_{k-1}}}\int_{\Delta_i^{(k)}}(1-\varphi_1(t))^\alpha\ldots(1-\varphi_{k-2}(t))^\alpha(1-\varphi_{k-1}(t))^\alpha dt=$$

$$=\sum_{i=1}^{2^{N_{k-1}}}(1-\varphi_1(t_i))^\alpha\ldots(1-\varphi_{k-2}(t_i))^\alpha\int_{\Delta_i^{(k)}}(1-\varphi_{k-1}(t))^\alpha dt,$$

where $t_i\in\Delta_i^{(k)}$. Then

$$\int_0^1(1-\varphi_1(t))^\alpha\ldots(1-\varphi_{k-1}(t))^\alpha dt=c\cdot\sum_{i=1}^{2^{N_{k-1}}}(1-\varphi_1(t_i))^\alpha\ldots(1-\varphi_{k-2}(t_i))^\alpha\int_{\Delta_i^{(k)}}dt=$$

$$=c\cdot\int_0^1(1-\varphi_1(t))^\alpha\ldots(1-\varphi_{k-1}(t))^\alpha dt=c\cdot c^{k-1}=c^k.$$

Now we present the product $\left(1-\varphi_0(t)\right)\cdots\left(1-\varphi_k(t)\right)$ in the form of the sum:

$$\left(1-a\,w_1\left(2^{N_0}t\right)\right)\ldots\left(1-a\,w_1\left(2^{N_{k-1}}t\right)\right)=\sum_{i=0}^{2^k-1}\hat{w}(i)\,\overline{w}_i(t),$$

where for each $i=q_1 2^0+\cdots+q_k 2^{k-1}$, $q_j=0,1$ we denote $\overline{w}_i(t)=w_{q_0 2^{N_0}+\cdots+q_{k-1}2^{N_{k-1}}}(t)$. It is easy to see that $\hat{w}(0)=1$ and $\left|\hat{w}(i)\right|\le a$ for $0<i<2^k$. Thus, for nonintersecting m and n we have $\overline{w}_m\cdot\overline{w}_n=\overline{w}_{m+n}$. By denoting $W(t)=-\sum_{i=1}^{2^k-1}\hat{w}(i)\,\overline{w}_i(t)$, we have $\int_0^1\left|1-W(t)\right|^\alpha dt<c^k$ and, therefore, $m\left\{t:\left|1-W(t)\right|\ge y\right\}\le y^{-\alpha}\int_0^1\left|1-W(t)\right|^\alpha dt<y^{-\alpha}\cdot c^k$.

The second statement of the Lemma is obvious from the construction of the polynomial.

The Lemma is proved.

Proof of the Main Lemma. Let us approximate the function f by polynomial P_1 so that $m\{t:|P_1-f|>\delta/2\}<\varepsilon/2$. We take a such that $0<a<\delta$, n such that $\left(\deg P_1+1\right)\dfrac{2^\alpha\left\|\hat{P}_1\right\|_1^\alpha}{\delta^\alpha}c^n<\dfrac{\varepsilon}{2}$ and take $y=\dfrac{\delta}{2\,\|\hat{P}_1\|_1}$.

We define the polynomial $P(t)=\sum_{k=0}^{\deg P_1}\hat{P}_1(k)w_k(t)W_k(t)$, where the polynomials $W_k(t)=W(t)\left\{N_1^{(k)},\ldots,N_n^{(k)}\right\}$ are chosen according to the Lemma, and the numbers $N_k^{(i)}$ are to be chosen later.

Now we put $M=\max\{\deg P_1,n\}$. For all $m\ge M$ we can choose the numbers $N_k^{(i)}$ from the set of numbers $N_k^{(i,m)}$ such that $\operatorname{spec}\{P\}\subset B_m$ for all $m\ge M$ and, therefore, $\operatorname{spec}\{P\}\subset\{\lambda_k\}_{k=1}^\infty$. Hence, we can choose numbers $N_k^{(i)}$ such that $\min\{\operatorname{spec}\{P\}\}>k_0$ for any given k_0. So the first and second statements of the Main Lemma are satisfied.

We have the following estimates: $\left|P-P_1\right|=\left|\sum_{k=0}^{\deg P_1}\hat{P}_1(k)w_k(t)\left(W_k(t)-1\right)\right|$,

$$m\left\{t:\left|P-P_1\right|\ge\sum_{k=0}^{\deg P_1}\left|\hat{P}_1(k)\right|y\right\}\le m\left\{t:\sum_{k=0}^{\deg P_1}\left|\hat{P}_1(k)\left(W_k(t)-1\right)\right|\ge\sum_{k=0}^{\deg P_1}\left|\hat{P}_1(k)\right|y\right\}\le$$

$$\le\sum_{k=0}^{\deg P_1}m\left\{t:\left|\hat{P}_1(k)\left(W_k(t)-1\right)\right|\ge\left|\hat{P}_1(k)\right|y\right\}=\sum_{k=0}^{\deg P_1}m\left\{t:\left|W_k(t)-1\right|\ge y\right\}\le\left(\deg P_1+1\right)y^{-\alpha}c^n.$$

Then $m\{t:|P-f|>\delta/2+\|\hat{P}_1\|_1\,y\}\le m\left\{t:|P-P_1|+|P_1-f|>\delta/2+\|\hat{P}_1\|_1\,y\right\}\le$

$$\le m\{t:|P-P_1|>\|\hat{P}_1\|_1\,y\}+m\{t:|P_1-f|>\delta/2\}<(\deg P_1+1)y^{-\alpha}c^n+\varepsilon/2<\varepsilon.$$

So, we have $m\{t:|P-f|>\delta\}<\varepsilon$.

The Main Lemma is proved.

Proof of the Theorem.

Theorem. There exists a series $\sum_{k=1}^{\infty} c_k w_{\lambda_k}(x)$ with $c_k \to 0$, which is universal in the usual sense for $L^0[0,1]$.

Proof. We denote by $\{f_n(x)\}_{n=1}^{\infty}$ the sequence of polynomials with rational coefficients and, applying successively the Main Lemma, we can choose a sequence of polynomials $Q_j(x)$ in subsystem of the Walsh system $Q_j(x) = \sum_{i=m_{j-1}}^{m_j-1} a_i w_{\lambda_i}(x)$, satisfying the following conditions:

1. $m\{x : | f_k(x) - \sum_{j=1}^{k} Q_j(x) | < 2^{-k}\} > 1 - 2^{-k}$;

2. $| a_i | < 2^{-j}$, for all $i \in [m_{j-1}, m_j)$.

Let $f(x) \in L^0[0,1]$. Let us choose a subsequence of polynomials $\{f_{\nu_m}\}$ such that $m\{| f(x) - f_{\nu_k}(x) | < 2^{-2k}\} > 1 - 2^{-k}$. Let $B_k = \left\{x : | f(x) - f_{\nu_k}(x) | < 2^{-2k}\right\}$, $E_k = \left\{x : | f_{\nu_k} - \sum_{j=1}^{\nu_k} Q_j(x) | < 2^{-\nu_k}\right\}$ and, finally, $E = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left(E_k \cap B_k\right)$. Obviously, $| E | = 1$. Then $\left| f(x) - \sum_{j=1}^{\nu_k} \left(\sum_{i=m_{j-1}}^{m_j-1} a_i w_{\lambda_i}(x) \right) \right| < 2^{-k}$ for all $x \in E_k \cap B_k$.

This means that $\lim_{k\to\infty} \sum_{i=1}^{\nu_k} a_i w_{\lambda_i}(x) = f(x)$ on E, i.e. $\sum_{i=1}^{\infty} a_i w_{\lambda_i}(x)$ is universal in the usual sense for $L^0[0,1]$ and $a_i \to 0$.

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REFERENCES

1. **Menshoff D.E.** // Mat. Sbornik, 1947, v. 20, p. 197 (in Russian).
2. **Kozlov W.Ya.** // Mat. Sbornik, 1950, v. 26, p. 351 (in Russian).
3. **Talalyan A.A.** // Izv. AN Arm. SSR. Matematika, 1957, v. 10, № 3, p. 17 (in Russian).
4. **Talalyan A.A.** // Uspehi Mat. Nauk, 1960, v. 15, № 5, p. 567–604 (in Russian).
5. **Grigorian M.G.** // Izv. NAN Armenii. Matematika, 2000, v.35, № 4, p. 23–29 (in Russian).
6. **Orlicz W.** // Bull. de l'Academie Polonaise des Sciences, 1927, v. 81, p. 117–125.
7. **Fichtengolz G.M.** A Course of Differential and Integral Calculus. V.II. M.: Nauka, 1996 (in Russian).
8. **Kozma G., Olevskii A.** // J. Anal. Math., 2001, v. 84, p. 361–393.
9. **Nalbandyan M.A.** // Izv. Vuzov. Matematika, 2009, v. 10, p. 51–62 (in Russian).
10. **Walsh J.L.** // Amer. J. Math., 1923, v. 45, p. 5–24.
11. **Paley R.E.A.C.** // Proc. London Math. Soc., 1932, v. 34, p. 241–279.