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Mathematics

ALMOST PERIODICITY IN SPECTRAL ANALYSIS REPRESENTATIONS INDUCED BY GENERALIZED SHIFT OPERATION

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The famous Theorem of Yu. Lubich allows us in the language of the almost periodicity to get the criterion of completeness of the eigenvectors of a Hermitian compact operator in a weakly complete Banach space. In this paper this result is strengthened for the representation generated by the operation of the generalized shift.

Keywords: representations, generalized shift operations, almost periodicity.

Let S be a compact Hermitian operator acting on a weakly complete Banach space X. Then $\exp(itS)$ is an isometric representation of the group $\mathbb{R}$ of the real numbers in the space X [1]. As it has been proved by Yu. I. Lubich, for the completeness in a weakly complete Banach space X of the eigenvectors system of the above mentioned operator S, it is necessary and sufficient that the function $\varphi(\exp(itS)\,x)$ be an almost periodic Bohr function on $\mathbb{R}$. Later a generalization of this results for the case of normal operators has been obtained in [2, 3]. In the present paper, using methods of [4, 5], we study the above-mentioned questions for the representations, generated by the generalized shift operator (g.s.o.) [6–8].

It is well known that the theory of almost periodic functions has been justified in [9], where is studied the integral equation

$$\lambda\varphi(s) = \lim_{T\to\infty} 1/2T \int_{-T}^{T} f(t-s)\varphi(t)dt, \tag{1}$$

where $f(t)$ is an almost periodical Bohr function.

Let $B(\mathbb{R})$ be the Banach algebra of complex-valued bounded functions on $\mathbb{R}$ with respect sup-norm. In connection with the equation (1), a more general integral equation is considered in [10–12]

$$\lambda\varphi(s) = M[Kf(s,t)\varphi(t)], \tag{2}$$

where $M(f)$ is the generalized limit (in Banach sense) [13, 14] in the algebra $B(\mathbb{R})$, that is $M(f) = \lim_{T\to\infty} 1/2T \int_{-T}^{T} f(t)dt$, and $K(s, t)$ is a continuous bounded function on $\mathbb{R}^2$.

Let $C_b(\mathbb{R}) \subset B(\mathbb{R})$ be the Banach subalgebra of all continuous bounded functions on $\mathbb{R}$. A family $\Phi \subset C_b(\mathbb{R})$ is called *equicontinuous* on $\mathbb{R}$, if all the

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functions from this family are equicountinuous on each finite interval, and if for any $\varepsilon > 0$ there exists a finite partition $\{E_1,\ldots,E_n\}$ of $\mathbb{R}$ such that for all $f \in \Phi$ and for arbitrary $x', x'' \in E_k$ ($k=1,\ldots,n$) the inequality $|f(x') - f(x'')| < \varepsilon$ holds.

The following analogy of Arzela's Theorem about precompactness in $C_b(\mathbb{R})$ [10] is true: the family $\Phi \subset C_b(\mathbb{R})$ is precompact in $C_b(\mathbb{R})$, iff Φ is bounded in $C_b(\mathbb{R})$ and is equicontinuous on $\mathbb{R}$.

Hence, as a consequence one can obtain that the operator

$$Af(s) = M[Kf(s,t)f(t)] \tag{3}$$

is compact, if the family $K(s, t)$ (where t is a parameter) is precompact in the $C_b(\mathbb{R})$. In this paper we will consider such kernels $K(s, t)$, for which the operator A defined by (3) is compact. First we introduce the g.s.o. in order to select almost periodic functions, connected with the integral equation (2) in the context of our study. We define the scalar product by the following formula: $\langle f,g\rangle = M[f(t)\overline{g}(t)]$, $f, g \in B(\mathbb{R})$, where $M[\varphi(t)]$, as above, is the generalized limit (in Banach sense).

The family of functions $f \in B(\mathbb{R})$ form a Bezikovich Hilbert space $B_M^2(\mathbb{R})$ with the norm $\|f\| = M[|f(t)^2|]$.

Let $\tau^{\mathbb{R}} = \{\tau^s\}_{s\in\mathbb{R}}$ be a family of a bounded operators, defined on the space of real variable functions $f(t)$ depending on a parameter s. Thus, for every function $f(t)$ we correspond a function of two variables $\tau^s f(t) = K(s,t)$.

We suppose that the family $\tau^{\mathbb{R}}$ satisfies the following axioms:

a) $\tau_t^0 = I$, $\tau_0^s = I$, where I is the identity operator;

b) the family of operators $\{\tau^s\}$ is bounded in a following sense: there exists a constant L independent of the function f and of the value of the parameter s such that $|\tau_t^s f(t)| \le L\sup|f(t)|$, $|(\tau_t^s)^{-1} f(t)| \le L\sup|f(t)|$;

c) τ^s is linear, i.e. $\tau_t^s[\alpha f(t) + \beta g(t)] = \alpha\tau_t^s f(t) + \beta_t^s g(t)$ for every $\alpha, \beta \in C$;

d) $\tau_t^s f(t)$ is a real function, if $f(t)$ is a real function;

e) the following family of conjugate operators $(\tau^{\mathbb{R}})^+$ are well defined by the formula $\langle \tau_t^s f(t), g(t)\rangle = \langle f(t), (\tau_t^s)^+, g(t)\rangle$.

A continuous function $f(t)$ is called τ-almost periodic, if the family of functions $\{\tau_t^s f(t)\}_s, \{(\tau_t^s)^+, f(t)\}_s$ are precompact on a real line in a uniform convergence sense. τ-Almost periodicity of the function f implies that $f \in C_b(\mathbb{R})$. We denote the space of all τ-almost periodic functions by $C_{AP}^{\tau}(\mathbb{R})$. This space form a closed subalgebra of the Banach algebra $C_b(\mathbb{R})$. We will suppose that the family of operators $\tau^{\mathbb{R}}$ satisfy the generalized shift conditions from [8] .We will also suppose that $\|(\tau_t^s)^+\| \le L$. Then $L^1(\mathbb{R},\mu)$ will become a symmetric Banach algebra, where

$$\left.\begin{aligned} \|f\|_{1,\tau} &= L\int_{\mathbb{R}} |f(t)|\,d\mu(t) \\ (f*g)_\tau(t) &= \int_{\mathbb{R}} (\tau_t^s)^+ f(t)g(s)\,d\mu(s) \end{aligned}\right\}. \tag{4}$$

We denote this algebra by $L^1_\tau(\mathbb{R})$ with involution defined by the formula

$$f^+(s) = \lim_{n\to\infty} \int_{\mathbb{R}} (\tau^s_t)^+ f(t) e_n(t) d\mu(t). \tag{5}$$

By adding the unit function to the algebra $L^1_\tau(\mathbb{R})$ we obtain a commutative symmetric Banach algebra $\mathfrak{R}_T(\mathbb{R})$ and the maximal ideals of the algebra $\mathfrak{R}_T(\mathbb{R})$, different from $L^1_\tau(\mathbb{R})$, are given by the formula

$$(\lambda I + f)(m) = \lambda + \int_{\mathbb{R}} f(t)\overline{\varphi(t,m)} d\mu(t), \tag{6}$$

where $\varphi(t,m)$ is a continuous function on $\mathbb{R} \times \mathfrak{M}_{\mathfrak{R}_\tau}$ satisfying

$$\tau^s_t \varphi(t,m) = \varphi(s,m) \cdot \varphi(t,m),$$

and $\mathfrak{M}_{\mathfrak{R}_\tau}$ is the space of maximal ideals of algebra $\mathfrak{R}_\tau$. Conversely, every such function defines by (6) a maximal ideal of algebra $\mathfrak{R}_\tau(\mathbb{R})$.

Let $m_0 = L^1_\tau(\mathbb{R})$ and $\mathfrak{M}'_{\mathfrak{R}_\tau} = \mathfrak{M}_{\mathfrak{R}_\tau} \setminus \{m_0\}$. Then there exists an unique measure υ on $\mathfrak{M}'_{\mathfrak{R}_\tau}$ such that

$$\left.\begin{aligned} \hat{f}(m) &= \int_{\mathbb{R}} f(t)\overline{\varphi(t,m)} d\mu(t) \\ f(t) &= \int_{\mathfrak{M}_{\mathfrak{R}_\tau}} \hat{f}(m)\varphi(t,m) d\nu(m) \end{aligned}\right\}, \text{ where } m \in \mathfrak{M}'_{\mathfrak{R}_\tau}.$$

Consider the integral equation $\lambda\varphi(s) = M_\tau[(\tau^s_t)^+ f(t)\varphi(t)]$, where $M_\tau(\cdot)$ is the invariant mean with respect to g.s.o. $\tau^{\mathbb{R}}$, as a generalized limit in Banach sense, and let the kernel $K(s,t) = (\tau^s_t)^+ f(t)$ be normal. Then the operator A defined by $A\varphi(s) = M_\tau[(\tau^s_t)^+ f(t)\varphi(t)]$, where $f \in C^\tau_{AP}(\mathbb{R})$, is a normal compact operator. Completing the algebra $C^\tau_{AP}(\mathbb{R})$ by a prehilbertian structure, defined by the scalar product $\langle f, g\rangle = M_\tau[f(t)\overline{g}(t)]$, we get the Hilbert space of τ-almost periodic Bezikovich functions on $\mathbb{R}$, which we denote by $B^2_\tau(\mathbb{R})$. It should be noted that the eigen-functions $\{\varphi(\cdot,m)\}_{m\in M_{\mathbb{R}_\tau}}$ are mutually orthogonal on the space $B^2_\tau(\mathbb{R})$, and the family of functions χ_m, where $\chi_m(t) = \varphi(t,m)/M_\tau[|\varphi(t,m)|^2]$, form an orthonormal basis in $B^2_\tau(\mathbb{R})$.

Let $T: \mathbb{R} \to \mathrm{Aut}(X)$ be an isometric representation of $\mathbb{R}$ in a Banach space X, generated by the g.s.o. τ^R, i.e. for all $t \in \mathbb{R}$ $\|T(t)\| = 1$. As it is accepted in the Banach representation theory, we will assume that the representation is strongly continuous.

We will call a vector $x \in X$ $(x \neq 0)$ an eigen-vector of the representation T, generated by the g.s.o. $\tau^{\mathbb{R}}$, if there exists a character $\chi \in \hat{\mathbb{R}}_\tau$ such that $T(s)x = \chi(s)x$ for each $s \in \mathbb{R}$.

Theorem 1. Let X be a weakly complete Banach space, and T is an isometric representation of $\mathbb{R}$ in the space X, generated by the g.s.o. $\tau^{\mathbb{R}}$. Then for the completeness of the system of eigen-vectors of representation T it is necessary and sufficient that $\varphi(T(t)x) \in C^\tau_{AP}(\mathbb{R})$ for all $x \in X$ and $\varphi \in X^*$.

Proof.

Necessity. Let for $\varepsilon > 0$ we have $\| x - \sum_{k=1}^{p} x_k \| < \varepsilon$, where $T(t)x_k = \chi_k(t)x_k$ for all $t \in \mathbb{R}$ and $k = 1,\dots,p$. Then $\| x - \sum_{k=1}^{p} \varphi(x_k)\chi_k(t) \| \le \varepsilon \| \varphi \|$, which implies that $\varphi(T(t)x) \in C_{AP}^{\tau}(\mathbb{R})$.

Sufficiency. Let for each $x \in X$, $\varphi \in X^*$ the function $\varphi(T(t)x) \in C_{AP}^{\tau}(\mathbb{R})$. We associate to the function $\varphi(T(t)x)$ with his Fourier–Bohr series in the space $B_{\tau}^{2}(\mathbb{R})$ that is $\varphi(T(t)x) \sim \sum_{\chi \in \hat{\mathbb{R}}_{\tau}} C_{\chi}\chi(t)$, where $C_{\chi} = M_{\tau}[\varphi(T(t)x)\overline{\chi}(t)] = \varphi(M_{\tau}[T(t)x\,\overline{\chi}(t)])$. A weak completeness of the space X implies the existence of the weak limit

$$P_{\chi}x = M_{\tau}[T(t)x \cdot \overline{\chi}(t)].$$

Let us prove that the family of operators $\{P_{\chi}\}_{\chi \in \hat{\mathbb{R}}_{\tau}}$ satisfy the following properties:

$$(1)\ \ P_{\chi} \cdot P_{\chi'} = \delta_{\chi\chi'} \cdot P_{\chi}; \qquad (2)\ \ T\,P_{\chi}x = \chi \cdot P_{\chi}x.$$

First, we have $\|P_{\chi}\| \le 1$, since T is an isometric representation of $\mathbb{R}$, generated by the g.s.o. $\tau^{\mathbb{R}}$. Let us prove the property (1):

$$P_{\chi}P_{\chi'}x = M_{\tau}[T(t)(P_{\chi'}x)\overline{\chi}(t)] = M_{\tau}[\chi(s)\overline{\chi'(s)}\ ] \cdot M_{\tau}[T(t)x\overline{\chi(t)}] = \delta_{\chi\chi'} \cdot P_{\chi}x.$$

Next, we check the property (2):

$$T(s)P_{\chi}x = T(s)M_{\tau}[T(t)x\overline{\chi(t)}] = \chi(s)M_{\tau}[T(t)x\overline{\chi}(t)] = \chi(s)P_{\chi}x.$$

Thus, the vector $x(\chi) = P_{\chi}x$ is an eigen-vector of the representation *T*.

Let $\varphi \in X^*$ be a functional satisfying $\varphi(x(\chi)) = 0$. Then we have $C_{\chi} = \varphi(P_{\chi}x) = \varphi(x(\chi)) = 0$. The condition $\varphi(T(t)x) = 0$ implies $\varphi(x) = 0$, and, hence, $\varphi = 0$, since the family $\hat{\mathbb{R}}_{\tau}$ forms an orthonormal basis in $B_{\tau}^{2}(\mathbb{R})$. Q.E.D.

Let $\chi \in \hat{\mathbb{R}}_{\tau}$, then the subspace $X_{\chi} = \{x \in X : T(t)x = \chi(t)x$ for each $t \in \mathbb{R}\}$ is called an invariant subspace of the representation *T*.

Theorem 2. Let *X* be a reflexive Banach space and *T* is an isometric representation of $\mathbb{R}$ in the space *X*, generated by the g.s.o. $\mathbb{R}$, all weighted subspaces X_{χ} for which are finite-dimensional. If the system of the eigen-vectors of *T* is complete in *X*, then there exists a complete system of functionals, which is biorthogonal to the union of bases for all weighted subspaces of the representation *T*. ▲

Proof. We consider the family of conjugate operators $\{T^*(t)\}_{t \in \mathbb{R}}$ in the space X^*. Since $\|T^*(t)\| = \|T(t)\| = 1$ and *T* is a representation generated by the g.s.o. $\tau^{\mathbb{R}}$, this generelized shift operations generates g.s.o. $\delta^{\mathbb{R}}$ such that the family $\{T^*(t)\}_{t \in \mathbb{R}}$ will be an isometric representation of $\mathbb{R}$ in the space X^*, generated by the g.s.o. $\delta^{\mathbb{R}}$ by the formula $(\delta_t^s T^*(t)\varphi)(x) = \varphi(\tau_t^s T(t)x)$. It is easy to see that $\delta_t^s T^*(t) = T^*(s)T^*(t)$.

By Theorem 1 $\varphi(T(t)x) \in C_{AP}^{\tau}(\mathbb{R})$ for each $x \in X$ and $\varphi \in X^*$, since the system of eigen-vectors of representation *T* is complete in the space *X*. For each

$\Phi \in X^{**}$ and $\varphi \in X^{*}$ we will have $\Phi(T^{*}(t)\varphi) = x(X^{*}(t)\varphi) = \varphi(T(t)x) \in C^{\tau}_{AP}(\mathbb{R})$, since any continuous functional on X^{*} is a vector $x \in X$.

Thus, the function $t \to T^{*}(t)\varphi$ is weakly continuous for all $\varphi \in X^{*}$. But then it is strongly continuous by De-Lu and Gliksberg's Theorem and, hence, it is a representation. By Theorem 1 the system of eigen-functionals of the representation T^{*} is complete in the space X^{*}. Let $X^{*}_{\chi} = \{\psi \in X^{*} : T^{*}(t)\psi = \chi(t)\psi \text{ for each } t \in \mathbb{R}\}$. If $\chi, \chi' \in \hat{\mathbb{R}}_{\tau}$, $\chi \neq \chi'$, and $\varphi \in X^{*}_{\chi'}$, $x \in X_{\chi}$, then $\langle x, \varphi \rangle = \varphi(x) = 0$. It can be proved as in [4] that $\dim X_{\chi} = \dim X^{*}_{\chi}$. Let $\{e_1,\ldots,e_n\}$ be a basis of X_{χ}. We choose some basis $\{\varphi_1,\ldots,\varphi_n\}$ of X^{*}_{χ} and reconstruct it to the biorthogonal basis to the system $\{e_1,\ldots,e_n\}$. Note that $\det(\varphi_j(e_k)) \neq 0$ and, consequently, the system of linear equations $\sum_{j=1}^{n} \alpha_j \varphi_j(e_k) = a_k \ (k = 1,\ldots,n)$ is solvable for each right-hand side of equations, confirming the possibility to transform the basis $\{\varphi_1,\ldots,\varphi_n\}$ to the biorthogonal basis to the system $\{e_1,\ldots,e_n\}$. The union of such bases on X^{*}_{χ} gives us a desired system of functionals. Q.E.D.

Remark. As a nontrivial example of a g.s.o. $\tau^{\mathbb{R}}$ one can take a family of operators $\{\tau^{s}\}_{s \in \mathbb{R}}$, connected with the differential operator $\mathcal{D}$: $\mathcal{D}u = u'' - \rho(x)u$, where $\rho(x)$ is an even entire function, which is real function on $\mathbb{R}$. For the functions from $C^{(2)}(\mathbb{R}^2)$ this family can be defined as the solution $u(s,t) = \tau^{s}_{t} f(t)$ of the partial differential equation $\partial^2 u / \partial s^2 - \rho(t)u = \partial^2 u / \partial t^2 - \rho(s)u$ with the initial conditions $u(s,0) = f(s)$, $\partial u / \partial t \,|_{t=0} = 0$. ▲

Note that the previous results remain valid, if instead of the real axis $\mathbb{R}$ one consider a local compact space.

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