PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2012, № 3, p. 21–28

Mathematics

ON THE GENERALIZED SAMPLE RANGE

V. K. SAGHATELYAN*

Chair of Actuarial Mathematics YSU, Armenia

A new statistic called "generalized sample range" is introduced and estimation of its expectation is provided. Maximum value of expectation of new range is determined. Characterization of corresponding distribution, which affords that maximum is obtained. The asymptotic behavior and limiting relations of the distribution and its characterization are considered.

***Keywords*:** order statistics, sample range, selection differential, characterization of distributions, bounds for ordered random variables, limiting distribution, generalization of sample range.

Introduction. In this article we are going to find boundaries and consider asymptotic behavior of a statistical quantity, which we called "generalized sample range". This statistical quantity actually is the generalization of regular sample range and in some sense selection differential. Analogous researches could be found for sample extremes, selection differential and for sums of order statistics and record values (see [1–3]). In particular, in [1] estimations for $EX_{n,n}$, $EX_{1,n}$ and EW_n, where $W_n = X_{n,n} - X_{1,n}$ and $X_{n,n}$ and $X_{1,n}$ are order statistics from the independent, identically distributed random variables sequence $X, X_1, X_2, \ldots$ with common continuous distribution function $F(x)$, can be founded. Also, similar results for so called selection differential $D(k,n) = \frac{1}{k}\sum_{i=n-k+1}^{n} X_{i,n}$ obtained by Nagaraja (see [2]). Further, in [3] one can find estimations of $EV_{n,m}$ and $ET_{n,m}$, where $V_{n,m} = X(N(m+1)) + \ldots + X(N(n))$, (here $N(n)$ is the number of all record values in $X_1, X_2, \ldots, X_n$), is the sum of all record values (see [4]), in $X_{m+1}, X_{m+2}, \ldots, X_n$ and $T_{n,m} = X_{m,m} + X_{m+1,m+1} + \ldots + X_{n,n}$.

Generalized Sample Range. First, let us define the generalized sample range. Let $X_1, X_2, \ldots, X_n$ be a sample from arbitrary distribution. The statistic $W_{k,n} = (X_{n,n} + \ldots + X_{n-k+1,n}) - (X_{1,n} + \ldots + X_{k,n})$ we will call generalized sample range.

* E-mail: v.saghatelyan@ysu.am

We are also going to consider its standardized version $\tilde{W}_{k,n} = \frac{1}{k} W_{k,n}$. It is obvious that for $k=1$ $W_{1,n} = W_n = X_{n,n} - X_{1,n}$, so, actually $W_{k,n}$ is the generalization of sample range W_n. The following theorem is true.

Theorem 1. Let $X_1, X_2, ...$ be independent, identically distributed random variables with common continuous distribution function $F(x)$. Moreover $EX_i = 0$, $DX_i = 1$ for any i. Then, for any $k = 1, 2, ...$, $n = 2, 3, ...$, $k \le n$, the following inequality holds

$$EW_{k,n} \le \mu_{k,n}, \tag{1}$$

where $\mu_{k,n} = knC_{n-1}^{k}\left(2\int\limits_{1/2}^{1}\left(\int\limits_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}dv\right)^2 du\right)^{1/2}$. Moreover, the equality in (1) attained only when inverse function for $F(x)$, which is defined as $G(u) = F^{-1}(x) = \inf\{x : F(x) \ge u\}$, has the form $G(u) = \frac{1}{\mu}\left(knC_{n-1}^{k}\int\limits_{1-u}^{u} z^{n-k-1}(1-z)^{k-1}dz\right)$,

where $\mu = knC_{n-1}^{k}\left(2\int\limits_{1/2}^{1}\left(\int\limits_{1-u}^{u} v^{n-k-1}\left(1-v\right)^{k-1}dv\right)^2 du\right)^{1/2}$.

Proof. The statistic $W_{k,n}$ can be represented as $W_{k,n} = T_2 - T_1$, where $T_2 = X_{n,n} + ... + X_{n-k+1,n}$ and $T_1 = X_{1,n} + ... + X_{k,n}$. It is easy to see that set of order statistics $X_{1,n}, ..., X_{k,n}$ in the case when $X_{k+1,n} = x$ can be viewed as a complete set of order statistics $Y_{1,k}^{(x)} \le Y_{2,k}^{(x)} \le ... \le Y_{k,k}^{(x)}$ built by the sequence $Y_1^{(x)}, Y_2^{(x)}, ..., Y_k^{(x)}$ of independent random variables with common conditional distribution function $F^{(x)}(u) = P\{Y^{(x)} < u\} = F(u)/F(x)$, $u \le x$. Consider

$$E(Y_{1,k}^{(x)} + Y_{2,k}^{(x)} + ... + Y_{k,k}^{(x)}) = E(Y_1^{(x)} + Y_2^{(x)} + ... + Y_k^{(x)}) = kEY^{(x)} = k/F(x)\int\limits_{-\infty}^{x} udF(u), \tag{2}$$

then

$$ET_1 = \int\limits_{-\infty}^{\infty}\left(k/F(x)\int\limits_{-\infty}^{x} udF(u)\right)dF_{k+1,n}(x), \tag{3}$$

where $F_{k+1,n}(x)$ is the distribution function of $X_{k+1,n}$. From (3), taking into account that $F_{k+1,n}(x) = \int\limits_{-\infty}^{x} nC_{n-1}^{k}(F(u))^k(1-F(u))^{n-k-1}dF(u)$ (see [2]), we obtain

$$ET_1 = knC_{n-1}^{k}\int\limits_{0}^{1} G(u)\left(\int\limits_{u}^{1} v^{k-1}\left(1-v\right)^{n-k-1}dv\right)du. \tag{4}$$

Similarly, since the value $X_{n-k,n} = x$ is fixed, the set of order statistics $X_{n-k+1,n}, ..., X_{n,n}$ can be represented as a complete set of k order statistics $W_{1,k}^{(x)}, W_{2,k}^{(x)}, ..., W_{k,k}^{(x)}$, which are built by sequence $W_1^{(x)}, W_2^{(x)}, ..., W_k^{(x)}$ of independent random variables with common conditional distribution function $F_{(x)}(u)=P\{W^{(x)}<u\}=(F(u)-F(x))/(1-F(x))$, $u\geq x$. Therefore, we get

$$E(W_{1,k}^{(x)}+W_{2,k}^{(x)}+...+W_{k,k}^{(x)})=E(W_1^{(x)}+W_2^{(x)}+...+W_k^{(x)})=$$
$$=kEW_1^{(x)}=k\int_x^{\infty}udF_{(x)}(u)=\frac{k}{1-F(x)}\int_x^{\infty}udF(u). \tag{5}$$

Hence, for the second summand ET_2 we have

$$ET_2=\int_{-\infty}^{\infty}\left(\frac{k}{1-F(x)}\int_x^{\infty}udF(u)\right)dF_{n-k,n}(x), \tag{6}$$

where $F_{n-k,n}(x)$ is the distribution function of $X_{n-k,n}$, and it is known that (see [1]) $F_{n-k,n}(x)=\int_{-\infty}^{x}nC_{n-1}^{n-k-1}(F(u))^{n-k-1}(1-F(u))^k dF(u)$. Thus, recalling that $G(u)=F^{-1}(x)$, from (6) we obtain

$$ET_2=knC_{n-1}^{n-k-1}\int_0^1\left(\int_v^1 G(u)du\right)v^{n-k-1}(1-v)^{k-1}dv=knC_{n-1}^{k}\int_0^1 G(u)\left(\int_0^u v^{n-k-1}(1-v)^{k-1}dv\right)du. \tag{7}$$

Using expressions (4) and (7), we finally get the following for generalized sample range:

$$EW_{k,n}=ET_2-ET_1=knC_{n-1}^{k}\int_0^1 G(u)\left(\int_0^u v^{n-k-1}(1-v)^{k-1}dv-\int_0^{1-u}v^{n-k-1}(1-v)^{k-1}dv\right)du. \tag{8}$$

Considering that our initial random variables $X_1, X_2,...$ have zero expectation and unit variance, we can assert that inverse function $G(x)$ satisfies conditions

$$\int_0^1 G(u)du=0,\ \int_0^1 G^2(u)du=1. \tag{*}$$

Taking into consideration, that we need to find such function $G(x)$, which attains maximum in (1), so, using techniques of classical calculus of variations, we have to find stationary values of (8) first. To do this, we must obtain the unconditional extremum for

$$EW_{k,n}-\lambda\int_0^1 G(u)du-\mu/2\int_0^1 G^2(u)du, \tag{9}$$

after which, using condition (*), we get the values of constants λ and μ. Consider

$$f(G(u))=G(u)\left(knC_{n-1}^{k}\left(\int_0^u v^{n-k-1}(1-v)^{k-1}dv-\int_0^{1-u}v^{n-k-1}(1-v)^{k-1}dv\right)\right)-\lambda G(u)-\frac{\mu}{2}G^2(u).$$

Solving the equation $f'(G)=0$, we obtain

$$knC_{n-1}^{k}\left(\int_0^u v^{n-k-1}(1-v)^{k-1}dv-\int_0^{1-u}v^{n-k-1}(1-v)^{k-1}dv\right)-\lambda-\frac{\mu}{2}G=0.$$

Hence,

$$G(u)=\frac{1}{\mu}\left[knC_{n-1}^{k}\left(\int_0^u v^{n-k-1}(1-v)^{k-1}dv-\int_0^{1-u}v^{n-k-1}(1-v)^{k-1}dv\right)-\lambda\right]. \tag{10}$$

Therefore, we saw that maximum in (9) is attainable and is reached when the function $G(u)$ has the form (10). To find the coefficients λ and μ, we use condition (*) for $G(x)$ described above. Thus, for λ we gain

$$\lambda = knC_{n-1}^{k}\int_{0}^{1}\left[\int_{0}^{u} v^{n-k-1}(1-v)^{k-1}\,dv - \int_{0}^{1-u} v^{n-k-1}(1-v)^{k-1}\,dv\right]du =$$

$$= knC_{n-1}^{k}\left(\int_{0}^{1/2}\left(\int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv\right)du + \int_{1/2}^{1}\left(\int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv\right)du\right).$$

Making change of variable $1-u=t$ in the last integral, we obtain

$$\lambda = knC_{n-1}^{k}\left(\int_{0}^{1/2}\left(\int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv\right)du - \int_{1/2}^{0}\left(\int_{t}^{1-t} v^{n-k-1}(1-v)^{k-1}\,dv\right)dt\right) =$$

$$= knC_{n-1}^{k}\left(\int_{0}^{1/2}\left(\int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv\right)du - \int_{0}^{1/2}\left(\int_{1-t}^{t} v^{n-k-1}(1-v)^{k-1}\,dv\right)dt\right) = 0.$$

It is obvious, that for $\lambda = 0$, the function $G(u)$ is symmetric with respect to 1/2.

$$G(1-u) = \frac{1}{\mu}knC_{n-1}^{k}\left(\int_{0}^{1-u} v^{n-k-1}(1-v)^{k-1}\,dv - \int_{0}^{u} v^{n-k-1}(1-v)^{k-1}\,dv\right) = -G(u).$$

Further, using (*) as above, we are obtaining an expression for μ

$$\mu^{2} = \left(knC_{n-1}^{k}\right)^{2}\left(\int_{0}^{1}\left(\int_{0}^{u} v^{n-k-1}(1-v)^{k-1}\,dv - \int_{0}^{1-u} v^{n-k-1}(1-v)^{k-1}\,dv\right)^{2} du\right). \qquad (11)$$

Let us consider the integral in the right-hand side of (11). It seems to be very lengthy, so, for the sake of simplification, we are formally introducing a notation $V = v^{n-k-1}(1-v)^{k-1}\,dv$. Thus, we have

$$\int_{0}^{1}\left(\int_{0}^{u} V - \int_{0}^{1-u} V\right)^{2} du = \int_{0}^{1/2}\left(\int_{0}^{u} V - \int_{0}^{1-u} V\right)^{2} du + \int_{1/2}^{1}\left(\int_{0}^{u} V - \int_{0}^{1-u} V\right)^{2} du = 2\int_{1/2}^{1}\left(\int_{0}^{u} V - \int_{0}^{1-u} V\right)^{2} du.$$

Using this last expression, our notation and (11), we get the following expression for μ:

$$\mu = knC_{n-1}^{k}\left(2\int_{1/2}^{1}\left(\int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv\right)^{2} du\right)^{1/2}. \qquad (12)$$

Recalling (10) and considering the results for λ and μ, we can write down the form of inverse function $G(u)$, which is determining the initial distribution function we needed:

$$G(u) = -G(1-u) = \frac{1}{\mu}\left[knC_{n-1}^{k}\left(\int_{1-u}^{u} z^{n-k-1}(1-z)^{k-1}\,dz\right)\right], \quad \frac{1}{2} \le u \le 1, \qquad (13)$$

where μ is defined by (12).

To finalize the proof we need to find the maximum of $EW_{k,n}$ and show that it is attained when the inverse function of initial distribution function is given by (13).

In terms of simplification, let us use our symbolic notation again $V = v^{n-k-1}(1-v)^{k-1}dv$, so, from (8) and symmetric property of $G(u)$ we obtain

$$EW_{k,n} = knC_{n-1}^{k}\int_0^1 G(u)\left(\int_{1-u}^{u} V\right)du = knC_{n-1}^{k}\left[-\int_0^{1/2} G(1-u)\left(\int_{1-u}^{u} V\right)du + \int_{1/2}^{1} G(u)\left(\int_{1-u}^{u} V\right)du\right] =$$

$$= knC_{n-1}^{k}\left[\int_1^{1/2} G(t)\left(\int_t^{1-t} V\right)dt + \int_{1/2}^{1} G(u)\left(\int_{1-u}^{u} V\right)du\right] = 2knC_{n-1}^{k}\int_{1/2}^{1} G(t)\left(\int_{1-t}^{t} V\right)dt \ge 0,$$

i.e. $EW_{k,n} = 2knC_{n-1}^{k}\int_{1/2}^{1} G(t)\left(\int_{1-t}^{t} v^{n-k-1}(1-v)^{k-1}dv\right)dt \ge 0$. Now we can use the well known Cauchy–Schwarz–Buniakowski inequality, which yields

$$EW_{k,n} \le 2knC_{n-1}^{k}\left[\int_{1/2}^{1} G^2(t)dt\right]^{1/2}\left[\int_{1/2}^{1}\left(\int_{1-t}^{t} v^{n-k-1}(1-v)^{k-1}dv\right)^2 dt\right]^{1/2}. \qquad (14)$$

It is easy to see that $\int_{1/2}^{1} G^2(t)dt = \frac{1}{2}\int_0^1 G^2(t)dt = \frac{1}{2}$. Really, symmetric property of $G(u)$ implies $G^2(u) = G^2(1-u)$, thus, $\int_0^{1/2} G^2(u)du = \int_{1/2}^{1} G^2(t)dt$, and using conditions (*) $\int_0^{1/2} G^2(u)du + \int_{1/2}^{1} G^2(u)du = 1$.

Returning back to (14), we can write

$$2knC_{n-1}^{k}\left[\int_{1/2}^{1}\left(\int_{1-t}^{t} v^{n-k-1}(1-v)^{k-1}dv\right)^2 dt\right]^{1/2} = \sqrt{2}\mu,$$

where μ is given by (12). And hence, $EW_{k,n} \le \sqrt{2}\mu\left(\frac{1}{2}\right)^{1/2} = \mu$. Further we must verify that maximum value μ is really attained when inverse of distribution function is given by (13). To make sure of it, we have to substitute (13) in (8):

$$EW_{k,n} = 2knC_{n-1}^{k}\int_{1/2}^{1} G(t)\left(\int_{1-t}^{t} v^{n-k-1}(1-v)^{k-1}dv\right)dt =$$

$$= 2knC_{n-1}^{k}\int_{1/2}^{1} G(t)\frac{\mu}{knC_{n-1}^{k}}G(t)dt = 2\mu\int_{1/2}^{1} G^2(t)dt = \mu.$$

Thus, the maximum value of $EW_{k,n}$ exists, it is equal to μ and attains when $G(u)$ is given by (13). The Theorem is proved.

Asymptotic Behavior of $EW_{k,n}$. We are going to consider limiting case of inequality for $E\frac{W_{k,n}}{n}$ when n tends to infinity and $k \sim \alpha n$, $0 < \alpha < 1$. Let U_1, U_2,… be a sequence of independent random variables distributed uniformly on [0,1], and let $U_{k,n}$ be the corresponding order statistics. It is known (see [5]) that distribution function of $U_{k,n}$ can be given by

$$P\{U_{k,n} \le x\} = \int_0^x \frac{n!}{(k-1)!(n-k)!} t^{k-1}(1-t)^{n-k}\,dt, \quad 0 \le x \le 1.$$

Therefore, for order statistic $U_{n-k,n-1}$ we have

$$P\{1-u \le U_{n-k,n-1} \le u\} = (n-1)C_{n-2}^{k-1} \int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv, \text{ if } u \ge 1/2, \quad (15)$$

and it is easy to obtain $EU_{n-k,n-1} = \frac{n-k}{n} = 1 - \frac{k}{n}$, $DU_{n-k,n-1} = \frac{(n-k)k}{n^2(n+1)}$. Because of $k \sim \alpha n$, $0 < \alpha < 1$, when n is very large, we have $EU_{n-k,n-1} \sim 1-\alpha$ and $DU_{n-k,n-1} \sim \frac{\alpha(1-\alpha)}{n}$, moreover, $U_{n-k,n-1} \xrightarrow{P} 1 - \frac{k}{n} = 1-\alpha$ when $n \to \infty$. Thus,

$$\begin{cases} (n-1)C_{n-2}^{k-1} \int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv = P\{1-u \le U_{n-k,n-1} \le u\} \xrightarrow[n\to\infty]{} 1, & \text{if } 1-\alpha \in [1-u, u], \\ (n-1)C_{n-2}^{k-1} \int_{1-u}^{u} v^{n-k-1}(1-v)^{k-1}\,dv = P\{1-u \le U_{n-k,n-1} \le u\} \xrightarrow[n\to\infty]{} 0, & \text{otherwise.} \end{cases}$$

Considering the definition of $W_{k,n}$, it can be assumed $0 < \alpha \le 1/2$ and because in (15) $1-u \le 1/2$, so, $1-u \le 1-\alpha$, that's why it remains to be seen whether $1-\alpha \le u$, i.e. the case when $u \in [1-\alpha, 1]$. Without loss of generality we may take $1/2 \le u \le 1$ and write down from (12):

$$\mu \sim \frac{knC_{n-1}^{k}\sqrt{2}}{(n-1)C_{n-2}^{k-1}} \left(\int_{1/2}^{1} \left(\int_{1-u}^{u} (n-1)C_{n-2}^{k-1} v^{n-k-1}(1-v)^{k-1}\,dv \right)^2 du \right)^{1/2} \sim$$
$$\sim \left[\int_{1-\alpha}^{1} 1^2\,du \right]^{1/2} \sim n\sqrt{2\alpha}. \quad (16)$$

From (16) the following estimation yields

$$E\frac{W_{k,n}}{n} = E\frac{W_{\alpha n,n}}{n} = E\frac{X_{n,n} + \ldots + X_{(1-\alpha)n+1,n}}{n} - E\frac{X_{1,n} + \ldots + X_{\alpha n,n}}{n} \le \sqrt{2\alpha}. \quad (17)$$

Further we are going to establish the distribution function, under which the maximum in (17) is attained. Recalling (13) and taking into account that $\frac{z^{n-k-1}(1-z)^{k-1}(n-1)!}{(n-k-1)!(k-1)!}$ is the density function of $U_{n-k,n-1}$ when $0 \le z \le 1$, we can write for $u \in [1/2,\ 1]$:

$$G(u) = \frac{1}{\mu}\left[\frac{knC_{n-1}^{k}(n-k-1)!(k-1)!}{(n-1)!} P\{1-u \le U_{n-k,n-1} \le u\} \right] =$$
$$= \frac{1}{\mu}\left[\frac{k!n!(n-k-1)!}{k!(n-k-1)!(n-1)!} P\{1-u \le U_{n-k,n-1} \le u\} \right] = \frac{n}{\mu} P\{1-u \le U_{n-k,n-1} \le u\}.$$

Let us notice that for $k \sim \alpha n$ and $\alpha \in (0, 1/2)$ the following limiting relation holds:

$$(U_{n-k,n-1}-(1-\alpha))\sqrt{n/\alpha(1-\alpha)}\sim N(0,1),$$

where $N(0,1)$ is the standard normal distribution function. Thus, when $n\to\infty$

$$U_{n-k,n-1}\sim(1-\alpha)+\xi\sqrt{\alpha(1-\alpha)/n}, \tag{18}$$

where ξ is random variable with standard normal distribution. Hence, from (18) we have $\begin{cases} P\{1-u\le U_{n-k,n-1}\le u\}\to 1 & \text{when } 1-\alpha\le u,\\ P\{1-u\le U_{n-k,n-1}\le u\}\to 0 & \text{when } 1-\alpha> u.\end{cases}$

Using this last expressions and (16), we get

$G(u)\sim\begin{cases} n/\mu\sim 1/\sqrt{2\alpha}, & \text{if } 1-\alpha\le u\le 1,\\ 0, & \text{if } 1/2\le u<\alpha\end{cases}$ on the other hand symmetry property yields $G(u)\sim\begin{cases} -1/\sqrt{2\alpha}, & \text{if } 0\le u\le\alpha,\\ 0, & \text{if } \alpha\le u\le 1/2\end{cases}$. Thus, we can write down the needed function as $G(u)=\begin{cases} -1/\sqrt{2\alpha} & \text{when } 0<u<\alpha,\\ 0 & \text{when } \alpha<u<1-\alpha,\\ 1/\sqrt{2\alpha} & \text{when } 1-\alpha<u\le 1.\end{cases}$

Notice that $G(0)=-1/\sqrt{2\alpha}$, $G(1)=1/\sqrt{2\alpha}$, and this inverse function determines the distribution function, which is concentrated in three points $-1/\sqrt{2\alpha}$, 0 and $1/\sqrt{2\alpha}$, with corresponding weights α, $1-2\alpha$ and α.

Summarizing what has been written above, we can say that the following result has been obtained. Let $W_{k,n}$ be the generalized sample range constructed by a sequence $X_1,X_2,...$ of independent random variables with common continuous distribution function with 0 expectation and unit variance. Also, let $F_1,F_2,...$ be a sequence of those distribution functions, which afford maximum in $E\left(\frac{W_{\alpha n,n}}{n}\right)$ for fixed values of $n=1,2,...$ and $0<\alpha<\frac{1}{2}$. The following theorem is true.

Theorem 2. When $n\to\infty$ the maximum value of $E\left(\frac{W_{\alpha n,n}}{n}\right)$ tends to $\sqrt{2\alpha}$, and the sequence of maximizing functions $F_1,F_2,...$ tends to a distribution function, which has symmetry property and is concentrated in three points $-\frac{1}{\sqrt{2\alpha}}$, 0 and $\frac{1}{\sqrt{2\alpha}}$, with corresponding weights α, $1-2\alpha$ and α.

Corollary 1. Notice that when $\alpha=\frac{1}{2}$ the limiting distribution function described in Theorem will be concentrated in two points -1 and 1 with equal weights $\frac{1}{2}$ for each.

Corollary 2. It is easy to see that similar results are true also for the components T_2 and T_1 of generalized sample range

$$E\frac{\left(X_{n,n}+\ldots+X_{(1-\alpha)n,n}\right)}{n}\xrightarrow[n\to\infty]{}\sqrt{\frac{\alpha}{2}} \text{ and } E\frac{\left(X_{1,n}+\ldots+X_{\alpha n,n}\right)}{n}\xrightarrow[n\to\infty]{}-\sqrt{\frac{\alpha}{2}}$$

relatively.

Corollary 3. It should be noted, that if we take $EX=a$ and $DX=\sigma^2$ instead of condition (*), then we can obtain for T_2 and T_1:

$$E\frac{\left(X_{n,n}+\ldots+X_{(1-\alpha)n,n}\right)}{n}\xrightarrow[n\to\infty]{}a+\sigma\sqrt{\frac{\alpha}{2}}$$

and

$$E\frac{\left(X_{1,n}+\ldots+X_{\alpha n,n}\right)}{n}\xrightarrow[n\to\infty]{}a-\sigma\sqrt{\frac{\alpha}{2}}.$$

Moreover, $E\frac{W_{\alpha n,n}}{n}\xrightarrow[n\to\infty]{}\sigma\sqrt{2\alpha}$, and this results correspond to the symmetric distribution concentrated in $-\frac{\sigma}{\sqrt{2\alpha}}$, 0 and $\frac{\sigma}{\sqrt{2\alpha}}$, with weights α, $1-2\alpha$ and α respectively.

Received 27.02.2012

REFERENCES

1. **Ahsanullah M., Nevzorov V.B.** Ordered Random Variables. New York: Nova Science Publ. Inc., 2001, p. 346.
2. **David H.** Order Statistics. Wiley–Interscience, 2003, p. 482.
3. **Nagaraja H.N.** Some Finite Sample Results for the Selection Differential. // Ann. Inst. Statist. Math., 1981, v. 33, p. 437–448.
4. **Nevzorov V.B.** Records: Mathematical Theory. // Rhode Island, AMS, Translations of Math. Monogr., 2001, v. 194, p. 167.
5. **Saghatelyan V.K.** Characterizations of Distributions by Moment Properties of Sums of Records and Successive Maxima. // Journal of Math. Sciences., 2009, v. 159, № 3, p. 358–362.