PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2012, № 3, p. 29–33

Mathematics

ON DEGENERATE NONSELF-ADJOINT DIFFERENTIAL EQUATIONS OF FOURTH ORDER

L. P. TEPOYAN *, H. S. GRIGORYAN

Chair of Differential Equations YSU, Armenia

We consider the degenerate nonself-adjoint differential equation of fourth order $Lu \equiv (t^{\alpha}u'')'' + au''' - pu'' + qu = f$, where $t \in (0;b)$, $0 \le \alpha \le 2$, $\alpha \ne 1$, a, p, q are the constant numbers and $a \ne 0$, $p > 0$, $f \in L_2(0,b)$. We prove that the statement of the Dirichlet problem for the above equation depends on the sign of the number a (Keldysh Teorem).

Keywords: Dirichlet problem, degenerate equations, weighted Sobolev spaces, spectral theory of linear operators.

1. Statement of the Problem. In the present paper we observe the Dirichlet problem for the following degenerate differential equation

$$Lu \equiv (t^{\alpha}u'')'' + au''' - pu'' + qu = f,$$

where $t \in (0, b)$, $0 \le \alpha \le 2$, $\alpha \ne 1$, a, p, q are the constant numbers and $a \ne 0$, $p > 0$, $f \in L_2(0, b)$.

We are interested in the nature of boundary conditions with respect to t, ensuring that the equation has unique solution for any $f \in L_2(0,b)$.

In the article [1] (there a or p equal to zero) has been proven that this conditions depend on the sign of a. This type of phenomenon was first noted by Keldysh in [2] for the degenerate elliptic equation of second order.

Dirichlet problem for the degenerate equation of second order have been considered in [3, 4] and for the degenerate equations of the fourth order in [5–7]. In this article we consider the case, when $a \ne 0$, $p > 0$, but with the restriction $0 \le \alpha \le 2$, $a \ne 0$.

2. Dirichlet Problem.

2.1. ***The Space*** $\dot{W}_{\alpha}^{2}$**.** Let $\dot{C}^2[0,b]$ be the set of twice continuously differentiable functions $u(t)$, defined on $[0,b]$ and satisfying the conditions

$$u(0) = u'(0) = u(b) = u'(b) = 0. \tag{2.1}$$

---

* E-mail: tepoianl@ysu.am

Let $\dot W^2_\alpha$, $\alpha \ge 0$, be the completion of $\dot C^2[0, b]$ in the norm

$$\| u \|^2_{\dot W^2_\alpha} = \int_0^b t^\alpha \, | u''(t) |^2 dt \qquad (2.2)$$

with the corresponding scalar product $\{u, v\} = (t^\alpha u'', v'')$, where $(\cdot, \cdot)$ is the scalar product in $L_2(0, b)$.

It is known (see, for instance, [8]) that the elements of $\dot W^2_\alpha$ are continuously differentiable functions on $[\varepsilon, b]$ for every $0 < \varepsilon < b$, whose first derivatives are absolutely continuous and $u(b) = u'(b) = 0$. Therefore, it is sufficient to explore properties of the elements from $\dot W^2_\alpha$ for small t.

Proposition 2.1. For every $u \in \dot W^2_\alpha$ close to $t = 0$ we have following estimates

$$| u(t) |^2 \le C_1 t^{3-\alpha} \| u \|^2_{\dot W^2_\alpha}, \ \alpha \ne 1, 3; \quad | u'(t) |^2 \le C_2 t^{1-\alpha} \| u \|^2_{\dot W^2_\alpha}, \ \alpha \ne 1. \qquad (2.3)$$

For $\alpha = 3$ the factor $t^{3-\alpha}$ should be replaced by $|\ln t|$; for $\alpha = 1$ the factor $t^{1-\alpha}$ by $|\ln t|$ and the factor $t^{3-\alpha}$ by $t^2 |\ln t|$.

It follows from relations (2.3) that for $\alpha < 1$ (weak degeneracy) the boundary conditions $u(0) = u'(0) = 0$ are "retained", while for $1 \le \alpha < 3$ (strong degeneracy) only the first condition is "retained". For $\alpha \ge 3$ both $u(0)$ and $u'(0)$ in general may be infinite. For example, if $u(t) = t^\beta \varphi(t)$, where $\varphi(t) \in C^2[0, b]$, $\varphi(b) = \varphi'(b) = 0$ and $\varphi(0) \ne 0$, then it is easy to check that for $\alpha > 3$ and $\frac{(3-\alpha)}{2} < \beta < 0$ the function $u(t)$ belongs to $\dot W^2_\alpha$, but $u\big|_{t=0}$ and $u'\big|_{t=0}$ do not exist [9].

Proposition 2.2. For every $1 \le \alpha \le 4$ we have a continuous embedding

$$\dot W^2_\alpha \to L_2(0, b), \qquad (2.4)$$

which for $1 \le \alpha < 4$ is compact.

Note that for the proof of the embedding (2.4) for $1 \le \alpha \le 4$, we use the first inequality of (2.3). For the case $\alpha = 4$, using the Hardy inequality (see [10]), we obtain the exact estimate $\| u \|^2_{L_2(0,b)} \le \frac{16}{9} \| u \|^2_{\dot W^2_4(0,b)}$.

It follows from Proposition 2.2, that for $1 \le \alpha \le 4$ we have the inequality

$$\| u \|_{L_2(0,b)} \le c \, \| u \|_{\dot W^2_\alpha}. \qquad (2.5)$$

Note that the embedding (2.4) for $\alpha = 4$ is not compact and for $\alpha > 4$ fails.

If we want to work within the space $L_2(0, b)$, we assume that the condition $0 \le \alpha \le 4$ is fulfilled. Moreover, we restrict ourselves to the case $0 \le \alpha \le 2$ to have $u' \in L_2(0, b)$ [1].

2.2. ***Non-Self-Adjoint Equation of the First Type***. In this section we consider Drichlet problem for the equation

$$Lu \equiv (t^\alpha u'')'' + au''' - pu'' + qu = f, \qquad (2.6)$$

where $t \in (0,b)$, $0 \le \alpha \le 2$, $\alpha \ne 1$, a, p, q are the constant numbers and $a > 0$, $p > 0$, $f \in L_2(0,b)$.

Let $\psi_h(t) \equiv 0$ for $0 \le t \le h$ and

$$\psi_h(t) = \begin{cases} h^{-3}(t-h)^2(5h-2t), & h < t < 2h, \\ 1, & 2h < t \le b. \end{cases}$$

Let $u_h(t) = u(t)\psi(t)$. Obviously, the function $u_h(t)$ belongs to the space $\dot{W}_\alpha^2$. We can prove that for every function $u \in \dot{W}_\alpha^2$ and $\alpha \ne 1, \alpha \ne 3$ the norm $\| u_h - u \|_{\dot{W}_\alpha^2}$ tends to zero by $h \to 0$ [1].

Definition 2.1. The function $u \in \dot{W}_\alpha^2$ is called a generalized solution of the Dirichlet problem for the Eq. (2.6), if for every $\theta \in \dot{W}_\alpha^2$, $0 < h < b$, holds the equality

$$(t^\alpha u'', \theta_h'') - a(u'', \theta_h') + p(u', \theta_h') + q(u, \theta_h) = (f, \theta_h)\,. \tag{2.7}$$

Note that in Definition 2.1 we cannot write (u'', θ') instead of (u'', θ_h'), since it in general does not exist.

Now consider a particular case of the Eq. (2.6) for $q = 0$

$$Mu \equiv (t^\alpha u'')'' + au''' - pu'' = f\,. \tag{2.8}$$

Theorem 2.1. The generalized solution of the Dirichlet problem for the Eq. (2.8) exists and is unique for every $f \in L_2(0, b)$, $0 \le \alpha \le 2$ and $\alpha \ne 1$.

Proof.

Existence. Let $1 < \alpha \le 2$. Denoting $u' = v$ and integrating the Eq. (2.8), we get $(t^\alpha v')' + av' - pv = F(t)$, where $F(t) = \int_0^t f(\tau)\,d\tau$.

Here is very important that for $0 \le \alpha \le 2$ the function $u' = v$ belong to $L_2(0,b)$. Now we can use the fact that this equation has unique solution in $\dot{W}_\alpha^2$ (completion of $\dot{C}^1[0,b]$ in the norm $\| u \|_{\dot{W}_\alpha^2}^2 = \int_0^b t^\alpha |u'(t)|^2 dt$). Moreover, the value $v(0)$ is finite and can be defined by $F(t)$, but cannot be given arbitrarily [3]. Since $u \in \dot{W}_\alpha^2$ and $0 \le \alpha \le 2$ consequently we have $u(0) = u(b) = 0$, thus, the equation $u' = v$ has unique solution. Now it is easy to verify that this function satisfies to the equality (2.7) (for $q = 0$) for every $\theta \in \dot{W}_\alpha^2$.

Uniqueness. Let $1 \le \alpha \le 2$. Suppose that $u \in \dot{W}_\alpha^2$ satisfies to the equality (2.7) (for $q = 0$) for every $\theta \in \dot{W}_\alpha^2$ and $f = 0$. We know that $u(0) = 0$ and $u'(0)$ is finite. If we put $\theta = u$ and pass to the limit (which exists) when $h \to 0$, we obtain that $(t^\alpha u'', u_h'') - a(u'', v_h') + p(u', u_h') \to \| u \|_{\dot{W}_\alpha^2}^2 + \frac{a}{2}|u'(0)|^2 + p\int_0^b |u'(t)|^2 dt = 0$.

Hence, we conclude that $u = 0$. □

Definition 2.2. We say that the function $u \in \dot{W}_\alpha^2$ belongs to the domain of definition $D(M)$ of the operator *M*, if for some $f \in L_2(0,b)$ is valid the equality (2.7) (for $q = 0$) for every $\theta \in \dot{W}_\alpha^2$. In this case we write $Mu = f$.

Thus, we get an operator $M : L_2(0,b) \to L_2(0,b)$.

Proposition 2.3. The inverse operator $M^{-1} : L_2(0,b) \to L_2(0,b)$ is compact for $0 \le \alpha \le 2$, $\alpha \ne 1$.

Proof. As consequence of Theorem 2.1, we get that inverse operator M^{-1} is defined on whole $L_2(0, b)$. At the same time, from these considerations it follows that for $u \in D(M)$ we have

$$(f,u) = \{u,u\}_\alpha + \frac{a}{2}|u'(0)|^2 + p\int_0^b |u'(t)|^2 dt.$$

Now, using the inequalities of Cauchy and (2.5), we conclude that

$$\| u \|_{\dot{W}_\alpha^2} \le \| Mu \|_{L_2(0,b)} .$$

Since the embedding (2.4) is compact, therefore, we get that the operator M^{-1} is compact. □

Similarly as in the proof of Proposition 2.3, it is easy to verify that the spectrum $\sigma(M)$ of the operator M lies in the right half-plane (see [1, 3]).

We can now consider the general equation, since the number $-q$ can be considered as a spectral parameter for the operator M . Hence, if $-q \notin \sigma(M)$ (in particular $q > 0$), we can state that the Eq. (2.6) is uniquely solvable for every $f \in L_2(0,b)$.

2.3. ***Nonself-Adjoint Equation of the Second Type.*** In this section we consider Dirichlet problem for the equation

$$Lu \equiv (t^\alpha v'')'' - av''' - pv'' + qv = g, \;\; a > 0, \; p > 0, \; g \in L_2(0,b). \tag{2.9}$$

First we investigate a particular case of the Eq. (2.9) for $q = 0$

$$Nv \equiv (t^\alpha v'')'' - av''' - pv'' = g. \tag{2.10}$$

Definition 2.3. We call $v \in L_2(0,b)$ the generalized solution of the Eq. (2.10), if for every $u \in D(M)$ we have

$$(Mu, v) = (u, g). \tag{2.11}$$

Definition 2.3, as usual, generates an operator $N : L_2(0, b) \to L_2(0, b)$.

Theorem 2.2. A generalized solution for the Eq. (2.10) exists and is unique for every $g \in L_2(0, b)$. The generalized solution fulfills to the Conditions 2.1.

Proof. The generalized solution for the Eq. (2.10) is unique, since the operator N is defined as adjoint to the operator M and the image $R(M)$ of the operator M coincides with the $L_2(0, b)$. The existence follows from the boundedness of the inverse operator M^{-1} [11, 12]. As in the proof of Theorem 2.1, we denote $v' = w$, and after integrating of the Eq. (2.10) we get $(t^\alpha w')' - aw' - pw = G(t)$, where $G(t) = \int_0^t g(\tau) d\tau + C$. We know [3] that this equation has unique generalized solution

$w \in \dot{W}_\alpha^2$ (see Section 2.2), which fulfills (in contrast to the Eq. (2.8)) to the conditions $w(0) = w(b) = 0$. Now we can uniquely solve the equation $v' = w$ using the conditions $v(0) = v(b) = 0$. To prove that the defined in this way function $v(t)$ is the desired solution, we take an element $u \in D(M)$, $Mu = f$. Then, for $h > 0$ we have $(t^\alpha u'', v_h'') - a(u'', v_h') + p(u', v_h') = (f, v_h)$. Passing to the limit in this equality when $h \to 0$ and integrating by parts, we get $(u, Nv) = (f, v)$, which is equivalent to the Eq. (2.11).

Note that the inverse operator N^{-1} also will be compact as adjoint to the operator M^{-1} and, therefore, the spectrum $\sigma(N)$ of the operator N is in the right half-plane.

Now we can observe the general Eq. (2.9) regarding the number $-p$ as spectral parameter for the operator N. As a result we get, that if $-p \notin \sigma(N)$, then the Eq. (2.9) has the unique solution, which fulfills the conditions (2.1).

Received 20.08.2012

REFERENCES

1. **Tepoyan L.** Degenerate Fourth-Order Diffeential-Operator Equations. // Differen. Uravn., 1987, v. 23(8), p. 1366–1376 (in Russian); English Transl. in Amer. Math. Soc., 1988, № 8.
2. **Keldysh M.V.** On Certain Cases of Degeneration of Equations of Elliptic Type on the Boundary of a Domain. // Doklady Akad. Nauk SSSR, 1951, v. 77, p. 181–183 (in Russian).
3. **Dezin A.A.** Degenerate Operator Equations. // Mat. Sbornik, M., 1982, v. 43(3), p. 287–298 (in Russian).
4. **Kornienko V.V.** On the Spectrum of Degenerate Operator Equations. // Mathem. Notes, 2000, v. 68(5), p. 576–587.
5. **Jaiani G.** On a Generalization of the Keldysh Theorem. // Georgian Mat. Journal, 1995, v. 2, № 3, p. 291–297.
6. **Jaiani G.** Theory of Cusped Euler-Bernoulli Beams and Kirchoff-Love Plates. // Lecture Notes of TICMI, 2002, v. 3.
7. **Makhover E.V.** Bending of a Plate of Variable Thickness with a Cusped Edge. // Scientific Notes of Leningrad State Ped. Institute, 1957, v. 17, № 2, p. 28–39 (in Russian).
8. **Burenkov V.I.** Sobolev Spaces on Domains. Teubner, 1998, 312 p.
9. **Tepoyan L.P.** On the Spectrum of a Degenerate Operator. // Izvestiya Natsionalnoi Akademii Nauk Armenii. Matematika, 2003, v. 38, № 5, p. 53–57.
10. **Kudryavtzev L.D.** On Equivalent Norms in the Weight Spaces. // Trudy Mat. Inst. AN SSSR, 1984, v. 170, p. 161–190 (in Russian).
11. **Dezin A.A.** Partial Degenerate Equations (An Introduction to a General Theory of Linear Boundary Vaiue Problems). Springer, 1987.
12. **Weidmann J.** Lineare Operatoren in Hilberträumen. Teil 1. Stuttgart: Grundlagen, Teubner Verlag, 2000.